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A note on distance and parallelism between two ARIMA processes

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Summary: The purpose of this paper is to analyze the relationship between a measure of dissimilarity between ARIMA models, the AR metric proposed by Piccolo (1984), and the condition of parallelism of two ARIMA processes. In particular, we show that two ARIMA processes are parallel if and only if this measure of dissimilarity is null. Thus we propose to utilize an estimator of the AR metric as test statistic in order to test for parallelism. We also conduct a small Monte-Carlo study to investigate the finite-sample properties of the test.

Keywords: AR metric, Parallelism, Time series.

1. Preliminaries and notation

In this paper we investigate the relationship between the notion of parallelism of two ARIMA processes and a measure of dissimilarity between ARIMA models proposed by Piccolo (1984, 1989, 1990). We remember that the process $\{W_t\}$is said to be an ARMA (p, q)process if $\{W_t\}$is stationary and if for every t

$$W_t - \phi_{w1}W_{t-1} - ... - \phi_{wp}W_p = \varepsilon_t + \theta_{w1}\varepsilon_{t-1} + ... + \theta_{wq}\varepsilon_{t-q} \qquad (1)$$

where $\varepsilon_t \sim WN(0, \sigma^2)$. The equation (1) can be written in the more compact form $\phi_w(B)W_t = \theta_w(B)\varepsilon_t$, where $\phi_w(z)$ and $\theta_w(z)$ are the

p^{th} and q^{th} degree polynomials in z (z is a complex variable) $\phi_w(z) = 1 - \phi_{w1}z - ... - \phi_{wp}z^p$ and $\theta_w(z) = 1 + \theta_{w1}z + ... + \theta_{wq}z^q$, with p, $q < +\infty$, and B is the backward shift operator defined by $B^j W_t = W_{t-j}, j = 0, \pm 1, ...$ An $ARMA(p,q)$ process defined by the equation (1) is said to be invertible if there exists a sequence of constants $\{\pi_j\}$ such that $\sum_{j=1}^{\infty}|\pi_j| < \infty$ and $W_t = \sum_{j=1}^{\infty}\pi_{wj}W_{t-j} + \varepsilon_t$ $t = 0, \pm 1, ...,$ where $\pi_w(z) = \phi_w(z)/\theta_w(z) = 1 - \pi_{w1}z - \pi_{w2}z^2 + ...$ Let $\{X_t\}$ and $\{Y_t\}$ be two invertible $ARMA$ processes. $\{X_t\}$ and $\{Y_t\}$ are parallel if and only if

$$\phi_x(z) = \phi_y(z) \tag{2}$$

and

$$\theta_x(z) = \theta_y(z). \tag{3}$$

This definition may be found in Steece and Wood (1985) and in Guo (1999). Note that two parallel processes with identical initial conditions are equal but for a scale factor, due to the different variance of the noise. The notion of parallelism can be extend to the ARIMA processes. Let $\{X_t\}$ and $\{Y_t\}$be two invertible ARIMA processes; X_t and Y_t can be represented in the following form:

$$\phi_x(B)(1-B)^{d_x}X_t = \theta_x(B)a_t, \tag{4}$$

$$\phi_y(B)(1-B)^{d_y}Y_t = \theta_y(B)b_t, \tag{5}$$

with $a_t \sim WN(0, \sigma_a^2)$, $b_t \sim WN(0, \sigma_b^2)$ and where $\phi_x(z)$, $\phi_y(z)$ and $\theta_y(z)$ are finite polynomials in z of degrees p_x, q_x, p_y and q_y respectively, d_x and d_y are two integers, $\phi_x(z) \neq 0$, $\theta_x(z) \neq 0$, $\phi_y(z) \neq 0$ and $\theta_y(z) \neq 0$ for $|z| \leq 1$. $\{X_t\}$ and $\{Y_t\}$ are parallel if and only if

$$\phi_x(z) = \phi_y(z), \tag{6}$$

$$\theta_x(z) = \theta_y(z) \tag{7}$$

and

$$d_x = d_y. \tag{8}$$

There are many practical situations in which it would be important to know whether two time series are parallel. Guo (1999), for example, point out that if two sets of data could be considered as coming from a common ARMA model (i. e. if the hypothesis of parallelism is accepted), one can obtain better estimates of the model parameters by pooling the data sets. Finally, we introduce a measure of dissimilarity between ARIMA models. Let F be the class of invertible ARIMA processes. Following Piccolo (1984, 1989, 1990), we define the distance between $\{X_t\}$, $\{Y_t\} \in F$ to be

$$d(X_t, Y_t) = \left[\sum_{i=1}^{\infty} (\pi_{xi} - \pi_{yi})^2 \right]^{1/2}. \tag{9}$$

The distance $d(X_t, Y_t)$ always exists and satisfies $\forall\ X_t, Y_t, Z_t \in F$ the following properties:

i. $d(X_t, Y_t) \geq 0$; ii. $d(X_t, Y_t) = d(Y_t, X_t)$; iii. $d(X_t, Y_t) \leq d(X_t, Z_t) + d(Z_t, Y_t)$.

We note that the distance between $X_t$and a white noise process ε_t (which is characterized by the null $(0, 0, ...)$ sequence) is

$$d(X_t, \varepsilon_t) = \left[\sum_{i=1}^{\infty} \pi_{xi}^2 \right]^{1/2}. \tag{10}$$

Thus, by property iii, we have that

$$d(X_t, Y_t) \leq \left[\sum_{i=1}^{\infty} \pi_{xi}^2 \right]^{1/2} + \left[\sum_{i=1}^{\infty} \pi_{yi}^2 \right]^{1/2}. \tag{11}$$

It is sufficiently clear that two ARIMA processes are parallel if and only if the distance between they is null. Although this fact seems to be familiar to many time series analysts, we cannot give reference to a proof. Therefore, we will offer a proof in the following.

2. Distance and parallelism

The question posed here is: which is the relationship between distance and parallelism? We note that, in general, the condition $d(X_t, Y_t) = 0$

does not imply the parallelism between $\{X_t\}$ and $\{Y_t\}$. Let us consider, for example, the following ARMA processes

$$(1-\alpha B)X_t = a_t \tag{12}$$

$$(1-\alpha B)(1-\beta B)Y_t = (1-\beta B)b_t \tag{13}$$

In this case $d(X_t, Y_t) = 0$ and $\{X_t\}$ and $\{Y_t\}$ are not parallel. Henceforth we will assume that the AR and MA polynomials of the ARMA models have no common factors. This assumption is crucial in order to establish the results of this section. First, we consider two invertible ARMA processes, $\{X_t\}$ and $\{Y_t\}$. We prove that $d(X_t, Y_t) = 0$ if and only if $\{X_t\}$ and $\{Y_t\}$ are parallel.

Theorem 1. *Let* $\{X_t\}$ *and* $\{Y_t\}$ *be two invertible* ARMA *processes.* $d(X_t, Y_t) = 0$ *if and only if* $\{X_t\}$ *and* $\{Y_t\}$ *are parallel.*

Proof. $(\Rightarrow)$ If $d(X_t, Y_t) = 0$, then $\pi_{xi} = \pi_{yi}$ $i = 1, 2, ...$ Hence $\pi_x(z) = \phi_x(z)/\theta_x(z) = \phi_y(z)/\theta_y(z) = \pi_y(z)$. Thus $\phi_x(z) = \frac{\phi_y(z)}{\theta_y(z)}\theta_x(z)$ and $\phi_y(z) = \frac{\phi_x(z)}{\theta_x(z)}\theta_y(z)$. If $\theta_x(z) \neq \theta_y(z)$, with $q_y \geq q_x$, and if the polynomials $\theta_x(z)$ and $\theta_y(z)$ have $0 \leq k < q_x$ common roots, we have that

$$\phi_x(z) = \frac{\phi_y(z)(1-\lambda_1^x z)(1-\lambda_2^x z)...(1-\lambda_{(q_x-k)}^x z)}{(1-\lambda_1^y z)(1-\lambda_2^y z)...(1-\lambda_{(q_y-k)}^y z)}.$$

Thus $\phi_x(z)$ is a polynomial of infinite order, but this is absurd since, by hypothesis, $q_x < +\infty$. On the other hand, if $\theta_x(z) \neq \theta_y(z)$, with $q_y < q_x$, and if the polynomials $\theta_x(z)$ and $\theta_y(z)$ have $0 \leq k \leq q_y$ common roots, we have that

$$\phi_y(z) = \frac{\phi_x(z)(1-\delta_1^y z)(1-\delta_2^y z)...(1-\delta_{(q_y-k)}^y z)}{(1-\delta_1^x z)(1-\delta_2^x z)...(1-\delta_{(q_x-k)}^x z)}.$$

Thus $\phi_y(z)$ is a polynomial of infinite order, but this is absurd since, by hypothesis, $q_y < +\infty$. It follows that

$$\theta_x(z) = \theta_y(z).$$

Since $\theta_x(z) = \theta_y(z)$, we have that $\phi_x(z) = \phi_y(z)$. Thus, we can conclude that $\{X_t\}$ and $\{Y_t\}$ are parallel. ($\Leftarrow$) If $\{X_t\}$ and $\{Y_t\}$ are parallel we

have that $\theta_x(z) = \theta_y(z)$ and $\phi_x(z) = \phi_y(z)$. Thus

$$\pi_x(z) = \phi_x(z)/\theta_x(z) = \phi_y(z)/\theta_y(z) = \pi_y(z),$$

that is $\pi_{xi} = \pi_{yi}$, $i = 1, 2, ...$ It follows that $d(X_t, Y_t) = 0$ The following lemma shows that if the distance between two ARIMA processes is null, then the orders of integration of the processes are the same.

Lemma 1. *Let* $\{X_t\}$ *and* $\{Y_t\}$ *be two invertible* ARIMA *processes, that is*

$$\phi_x(B)(1-B)^{d_x} X_t = \theta_x(B) a_t, \quad a_t \sim WN(0, \sigma_a^2),$$

$$\phi_y(B)(1-B)^{d_y} Y_t = \theta_y(B) b_t, \quad b_t \sim WN(0, \sigma_b^2),$$

where $\phi_x(z)$, $\theta_x(z)$, $\phi_y(z)$ *and* $\theta_y(z)$ *are finite polynomials in* z *of degrees* p_x, q_x, p_y *and* q_y *respectively,* d_x *and* d_y *are two integers,* $\phi_x(z) \neq 0$, $\theta_x(z) \neq 0$, $\phi_y(z) \neq 0$ *and* $\theta_y(z) \neq 0$ *for* $|z| \leq 1$. *If* $d(X_t, Y_t) = 0$, *then* $d_x = d_y$.

Proof. If $d(X_t, Y_t) = 0$, then $\pi_{xi} = \pi_{yi}$ $i = 1, 2, ...$ Since

$$1 - \pi_{x1} z - \pi_{x2} z^2 - ... = \frac{\phi_x(z)(1-z)^{d_x}}{\theta_x(z)}$$

and

$$1 - \pi_{y1} z - \pi_{y2} z^2 - ... = \frac{\phi_y(z)(1-z)^{d_y}}{\theta_y(z)},$$

we have that

$$\frac{\phi_x(z)(1-z)^{d_x}}{\theta_x(z)} = \frac{\phi_y(z)(1-z)^{d_y}}{\theta_y(z)},$$

that is

$$\phi_x(z) = \frac{\theta_x(z)\phi_y(z)}{\theta_y(z)} \frac{(1-z)^{d_y}}{(1-z)^{d_x}}$$

Let us assume that $d_x \neq d_y$. In particular, without loss of generality, we can set $d_y = d_x + k$ (k integer). We have that

$$\phi_x(z) = \frac{\theta_1(z)\phi_y(z)}{\theta_y(z)}(1-z)^k.$$

Thus $\phi_x(z)$ has k unit roots. This result is absurd since, by assumption, $\phi_x(z) \neq 0$, for $|z| \leq 1$; it follows that $d_x = d_y$. The following theorem

extends the previous result to the ARIMA processes.

Theorem 2. *Let $\{X_t\}$ and $\{Y_t\}$ be two invertible* ARIMA *processes, that is*

$$\phi_x(B)(1-B)^{d_x}X_t = \theta_x(B)a_t, \quad a_t \sim WN(0,\sigma_a^2),$$
$$\phi_y(B)(1-B)^{d_y}Y_t = \theta_y(B)b_t, \quad b_t \sim WN(0,\sigma_b^2),$$

where $\phi_x(z)$, $\theta_x(z)$, $\phi_y(z)$ and $\theta_y(z)$ are finite polynomials in z of degrees p_x, q_x, p_y and q_y respectively, d_x and d_y are two integers, $\phi_x(z) \neq 0$, $\theta_x(z) \neq 0$, $\phi_y(z) \neq 0$ and $\theta_y(z) \neq 0$ for $|z| \leq 1$. $d(X_t, Y_t) = 0$ if and only if $\{X_t\}$ and $\{Y_t\}$ are parallel.

Proof. ($\Rightarrow$) If $d(X_t, Y_t) = 0$, then, by lemma 1, $d_x = d_y$ and

$$\frac{\phi_x(z)}{\theta_x(z)} = \frac{\phi_y(z)}{\theta_y(z)}.$$

Thus the distance $d(V_t, W_t)$ between the ARMA processes $V_t = (1-B)^{d_x}X_t$ and $W_t = (1-B)^{d_y}Y_t$ is zero. By theorem 1, it follows that the ARMA processes $\{V_t\}$ and $\{W_t\}$ are parallel. Thus we can conclude that $\phi_x(z) = \phi_y(z)$, $\theta_x(z) = \theta_y(z)$ and $d_x = d_y$,that is $\{X_t\}$ and $\{Y_t\}$ are parallel. ($\Leftarrow$) If $\{X_t\}$ and $\{Y_t\}$ are parallel we have that $\phi_x(z) = \phi_y(z)$,

$\theta_x(z) = \theta_y(z)$ and $d_x = d_y$. Thus

$$\pi_x(z) = \frac{\phi_x(z)(1-z)^{d_x}}{\theta_x(z)} = \frac{\phi_y(z)(1-z)^{d_y}}{\theta_y(z)} = \pi_y(z),$$

that is $\pi_{xi} = \pi_{yi}$, $i = 1, 2, ...$ It follows that $d(X_t, Y_t) = 0$

3. Testing for parallelism

Given the invertible ARIMA processes, $\{X_t\}$ and $\{Y_t\}$,

$$\phi_x(B)(1-B)^{d_x}X_t = \theta_x(B)a_t, \quad a_t \sim WN(0,\sigma_a^2),$$

$$\phi_y(B)(1-B)^{d_y}Y_t = \theta_y(B)b_t, \quad b_t \sim WN(0,\sigma_b^2),$$

the null hypothesis for a test for parallelism can be expressed as

$$H_0: \ \ \phi_x(z) = \phi_y(z), \ \ \theta_x(z) = \theta_y(z) \text{ and } d_x = d_y.$$

In this section we present a test statistics that can be used to decide whether or not the null hypothesis of parallelism is true. Considering a convenient finite approximation of the $AR(\infty)$ representation of the processes, $\{X_t\}$ and $\{Y_t\}$, and using suitable estimates of π_{xj} and π_{yj}, say $\widetilde{\pi}_{xj}$ and $\widetilde{\pi}_{yj}$, we obtain the distance estimator:

$$\widetilde{d}_k = \left[\sum_{i=1}^{\infty}(\widetilde{\pi}_{xi} - \widetilde{\pi}_{yi})^2\right]^{1/2}. \tag{14}$$

By theorem 2, we know the condition of parallelism between two invertible ARIMA processes, $\{X_t\}$ and $\{Y_t\}$ is equivalent to the condition $d(X_t, Y_t) = 0$. Thus to test the null hypothesis (2) we can use the statistic $\widetilde{d}_k^2$. We reject H_0 when $\widetilde{d}_k^2$ is sufficiently large. Piccolo (1989) shows that the asymptotic distribution of $\widetilde{d}_k^2$ is a linear combination of independent chi-square variables and Corduas (1996) approximates it with a single chi-square distribution.

4. A simulation study

To analyze the finite-sample properties of the test presented in the previous section, a small Monte Carlo sampling experiment is conducted. The study considers two different data generating processes (DGPs): first X_t and Y_t are two AR(1) independent stationary processes $(1-\phi_x B)X_t = a_t$, $(1-\phi_y B)Y_t = b_t$, with $\phi_x = 0.5$, $\phi_y = 0.5,\ 0.3,\ 0.1$, (DGP I), and second, X_t and Y_t are two MA(1) independent stationary processes $X_t = (1 - \theta_x B)a_t$, $Y_t = (1 - \theta_y B)b_t$, with $\theta_x = 0.5$, $\theta_y = 0.5$, 0.3, 0.1, (DGP II). We use RNDN function in GAUSS for Windows NT/95 Version 3.2.38, programming language, in order to generate the pseudo-normal variates a_t and b_t. Our Monte Carlo experiments are all based on 10,000 simulations. In each experiments we use $T = 50$, 100, 200 observations. In order to carry out this simulation study we have made of the DIST GAUSS-program presented in Corduas (2000). The results for DGP I and DGPII are reported in Tables 1 and 2, respectively.

Table 1. Percentage of rejections of the null at nominal 5% level.

ϕ_x ϕ_y	$T = 50$	$T = 100$	$T = 200$
.5 .5	4.97	5.29	5.06
.5 .3	18.43	33.00	57.75
.5 .1	54.50	83.26	98.69

Table 2. Percentage of rejections of the null at nominal 5% level.

θ_x θ_y	$T = 50$	$T = 100$	$T = 200$
.5 .5	3.44	3.53	3.05
.5 .3	10.49	18.96	35.02
.5 .1	39.04	67.67	93.74

For the AR case, the results from our Monte Carlo study suggest that: i. The effective size of the tests is close to the nominal size fixed at 5%. ii. The series of length 50 are not long enough for the test statistics to display their asymptotic behavior. iii. For sample sizes of 100 or more the test $\widetilde{d}_k^2$ has quite good size performance and a good power. The performances of the test are less satisfactory in the MA case.

5. Conclusions

In this paper we have investigated the relationship between the notion of parallelism of two ARIMA processes and the measure of dissimilarity between ARIMA models proposed by Piccolo (1984, 1989, 1990). In particular, we have shown that two ARIMA processes are parallel if and only if this measure of dissimilarity is null. Thus we have utilized an estimator of the AR metric as test statistic in order to testing for parallelism. We have also conducted a small Monte-Carlo study to investigate the finite-sample properties of the test. Our results suggest that for moderately large sample sizes such as 100 the performance of the test seems acceptable.

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