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Classifying the markets volatility with ARMA distance measures

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Summary: The financial time series are often characterized by similar volatility structures. The selection of series having a similar behavior could be important for the analysis of the transmission mechanisms of volatility and to forecast the time series, using the series with more similar structure. In this paper a metrics is developed in order to measure the distance between two GARCH models, extending well known results developed for the ARMA models. The statistic used to calculate it follows known distributions, so that it can be adopted as a test procedure. This tool can be used to develop an agglomerative algorithm in order to detect clusters of homogeneous series.

Keywords: GARCH Models, Clusters, Agglomerative Algorithm.

1. Introduction

The financial time series are generally subject to co-movements and similar volatility structures, due to the strong influence among financial markets (see, for example, Bollerslev et al., 1994). Generally, "trouble" and "quiet" periods are transmitted from a market to another, but some markets absorb more these effects. The classification of financial time series in homogeneous clusters for similar volatility structures could be an important purpose for the financial analysts, also because movements in a given time series could help to forecast the movements of a similar time series.

In this paper we extend the distance measure proposed in the seminal paper of Piccolo (1984) and then by Piccolo (1989,1990) for AR models to the case of the GARCH (*Generalized AutoRegressive Conditional Heteroskedasticity*) family. As stressed by Otranto and Triacca (2002), this distance compares the stochastic properties of a couple of series, or, in other words, the differences between the two data generating processes. In practice, the basic idea is that the estimation of GARCH models provides the statistical structure of the financial time series, so that the comparison of the models underlying the data generating processes is equivalent to compare the volatility structures of each series. The extension of this distance to the GARCH models is easy, considering the correspondence between GARCH and ARMA processes; in practice we express the residuals of a GARCH model in ARMA form and then we use, as in Otranto and Triacca (2002), the representation of ARMA models in AR terms (see, for example, Brockwell and Davis, 1996) to apply the distance measure. This representation provides a formulation of the distance measure as a function of the GARCH parameters. In addition, the statistic calculated to measure the distance follows a known asymptotic distribution, so that it is possible to use it as a test procedure. If we select the series having distance not significantly different from zero, it is possible to cluster the homogeneous series. In particular, we develop an agglomerative algorithm, based on the distance measure proposed and on the results of the statistical test. The methodology is applied to classify the series of the returns of the main financial markets. In the next section we will illustrate the instruments adopted to explicit the distance measure, studying the behavior of the distance proposed; we will pay a particular attention to the GARCH(1,1) model, which is the most popular model adopted for financial time series. Section 3 is devoted to the explanation of the use of this distance in classifying the volatility of markets; we illustrate the agglomerative algorithm and show an application of the procedure to nine stock exchange indices. Final remarks follow. In the final appendix, there is a report of some details on the AR metrics proposed by Piccolo (1984, 1989, 1990).

2. Distance between GARCH models

The GARCH family is very popular in time series analysis and it is composed of a large set of models, which can represent different possible characteristics of financial time series; for a review of these models and their applications see Bollerslev et al. (1992) and Bollerslev et al. (1994).

For our purpose, we consider two time series following the models ($t = 1, ..., T$):

$$\begin{aligned} y_{1,t} &= \mu_1 + \varepsilon_{1,t}, \\ y_{2,t} &= \mu_2 + \varepsilon_{2,t}; \end{aligned} \tag{1}$$

where $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are mean zero heteroskedastic independent disturbances. In other terms, the two series have a constant mean, whereas the variances are time-varying. We suppose that the conditional variances $h_{1,t}$ and $h_{2,t}$ follow two different and independent GARCH(1,1) structures:

$$\begin{aligned} Var(y_{1,t}|I_{1,t-1}) &= h_{1,t} = \gamma_1 + \alpha_1 \varepsilon^2_{1,t-1} + \beta_1 h_{1,t-1} \\ Var(y_{2,t}|I_{2,t-1}) &= h_{2,t} = \gamma_2 + \alpha_2 \varepsilon^2_{2,t-1} + \beta_2 h_{2,t-1} \end{aligned} \tag{2}$$

where $I_{1,t}$ and $I_{2,t}$ represent the information available at time t and $\gamma_i > 0$, $0 < \alpha_i < 1$, $0 < \beta_i < 1$, $(\alpha_i + \beta_i) < 1$ ($i = 1, 2$). This is a typical representation for financial time series.

Equation (2) implies that the squared residuals follow ARMA(1,1) processes:

$$\varepsilon^2_{i,t} = \gamma_i + (\alpha_i + \beta_i)\,\varepsilon^2_{i,t-1} - \beta_i\left(\varepsilon^2_{i,t-1} - h_{i,t-1}\right) + \left(\varepsilon^2_{i,t} - h_{i,t}\right),\ i = 1, 2 \tag{3}$$

where $\varepsilon^2_{i,t} - h_{i,t}$ are mean zero errors, uncorrelated with past information. Substituting in (3) the errors with their ARMA(1,1) expression, we obtain the AR(∞)representation:

$$\varepsilon^2_{i,t} = \frac{\gamma_i}{1 - \beta_i} + \alpha_i \sum_{j=1}^{\infty} \beta_i^{j-1} \varepsilon^2_{i,t-j} + \left(\varepsilon^2_{i,t} - h_{i,t}\right). \tag{4}$$

In this form, the two GARCH(1,1) models can be compared in terms of the distance measure proposed by Piccolo (1984, 1989, 1990), explained in the final appendix. In particular, recalling that the general form of this metrics is:

$$\left[\sum_{j=1}^{\infty}(\pi_{1j}-\pi_{2j})^2\right]^{1/2} \tag{5}$$

where π_{1j} and π_{2j} are the autoregressive coefficients of two AR processes, using (4), we can express the distance between two GARCH(1,1) models as:

$$d=\left[\sum_{j=0}^{\infty}\left(\alpha_1\beta_1^j-\alpha_2\beta_2^j\right)^2\right]^{1/2}$$

Developing the expression in square brackets:

$$d=\left[\alpha_1^2\sum_{j=0}^{\infty}\beta_1^{2j}+\alpha_2^2\sum_{j=0}^{\infty}\beta_2^{2j}-2\alpha_1\alpha_2\sum_{j=0}^{\infty}(\beta_1\beta_2)^j\right]^{1/2}= \tag{6}$$
$$=\left[\frac{\alpha_1^2}{1-\beta_1^2}+\frac{\alpha_2^2}{1-\beta_2^2}-\frac{2\alpha_1\alpha_2}{1-\beta_1\beta_2}\right]^{1/2}$$

Note that in the previous developments the constant $\gamma_i/(1-\beta_i)$ was not considered; in effect, it does not affect the dynamics of the volatility of the two series, expressed by the autoregressive terms.

It is very simple to extend that to more general cases; in fact, the GARCH(p,q) model (Bollerslev, 1986):

$$h_t=\gamma+\alpha_1\varepsilon_{t-1}^2...+\alpha_p\varepsilon_{t-p}^2+\beta_1h_{t-1}+...+\beta_qh_{t-q}$$

corresponds to the ARMA(p*,q) model, with p* $=\max(p,q)$:

$$\varepsilon_t^2=\gamma+(\alpha_1+\beta_1)\varepsilon_{t-1}^2...+(\alpha_{p^*}+\beta_{p^*})\varepsilon_{t-p^*}^2-\beta_1(\varepsilon_{t-1}^2-h_{t-1})-...-$$
$$-\beta_q(\varepsilon_{t-q}^2-h_{t-q})+(\varepsilon_t^2-h_t).$$

Of course, if $p>q$, we put $\beta_{q+1}=...=\beta_p=0$; if $q>p$, then $\alpha_{p+1}=...=\alpha_q=0$.

The ARCH(p) model (Engle, 1982):

$$h_t=\gamma+\alpha_1\varepsilon_{t-1}^2...+\alpha_p\varepsilon_{t-p}^2$$

corresponds to the AR(p) model:

$$\varepsilon_t^2 = \gamma + \alpha_1 \varepsilon_{t-1}^2 ... + \alpha_p \varepsilon_{t-p}^2 + (\varepsilon_t^2 - h_t);$$

the IGARCH(1,1) model (Engle and Bollerslev, 1986):

$$h_t = \gamma + (1 - \beta_1)\varepsilon_{t-1}^2 + \beta_1 h_{t-1}$$

corresponds to the IMA(1,1) model:

$$(\varepsilon_t^2 - \varepsilon_{t-1}^2) = \gamma - \beta_1(\varepsilon_{t-1}^2 - h_{t-1}) + (\varepsilon_t^2 - h_t);$$

and so on.

In general, indicating with ϕ_k the generic AR coefficient and θ_j the generic MA coefficient of an ARMA model, we have:

$$\begin{aligned} \phi_k &= (\alpha_k + \beta_k), \\ \theta_j &= -\beta_j. \end{aligned} \quad (7)$$

To apply (5) we need the AR representation of the ARMA model; following Brockwell and Davis (1996), the iterative formula:

$$\pi_k + \sum_{j=1}^{q} \theta_j \pi_{k-j} = -\phi_k, \quad k = 0, 1, ...$$

with $\phi_0 = 1$, can be applied. For the GARCH case, the previous relationship is equivalent to:

$$\pi_k = -(\alpha_k + \beta_k) + \sum_{j=1}^{q} \beta_j \pi_{k-j} = -\alpha_k + \sum_{j=1}^{q-1} \beta_j \pi_{k-j}. \quad (8)$$

Using (8) it is possible, applying (5), to compare every couple of GARCH models, not necessarily of the same order. In the remain of the work we will refer to GARCH(1,1) models, which are the most popular models for financial time series and for which the simple form (6) can be applied.

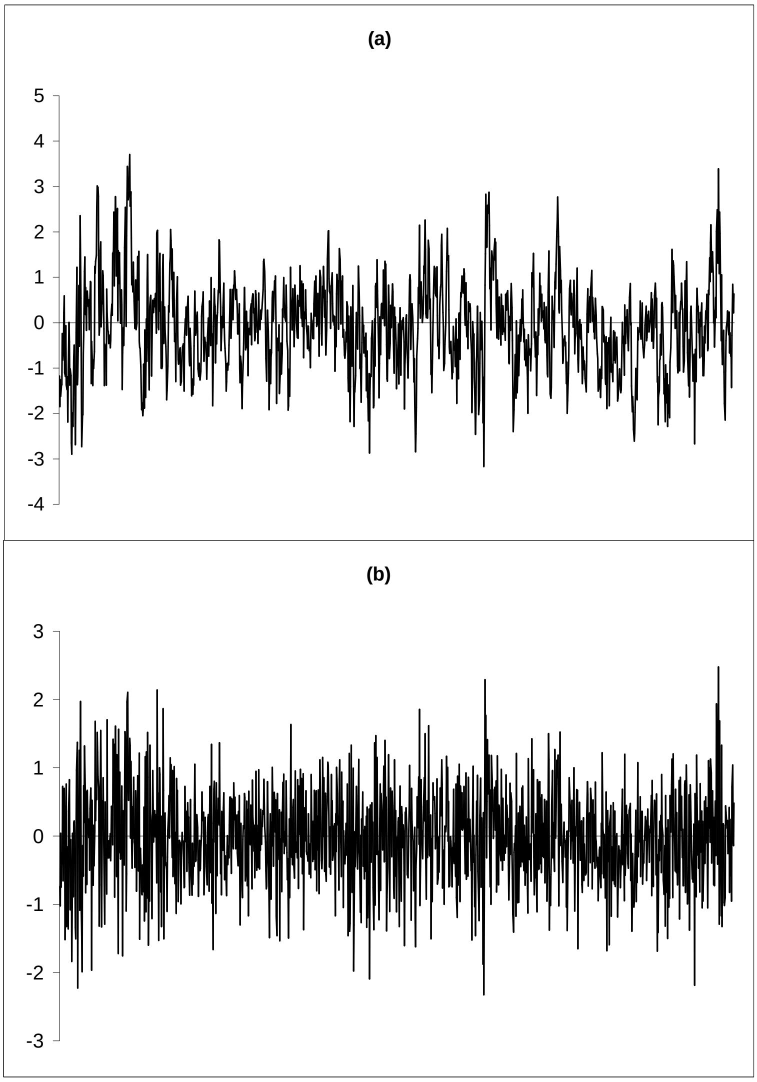

Figure 1. Simulated time series with the same GARCH(1,1) structure for the variance; picture (a): the level follows an AR(1) model; picture (b): the level follows an AR(0) model.

Note that, for the sake of simplicity, it was supposed in (1) that the levels of the series do not follow ARIMA structures. This assumption is often made in the study of returns, but we can suppose particular dynamics for $y_{1,t}$ and $y_{2,t}$. It is interesting to underline that series apparently different in their dynamics can have the same volatility structure; for example, in Figure 1 two simulated series of length 1352[1] with disturbances following the same GARCH(1,1) model are showed; but the level of the first one follows an AR(1) model (with AR coefficient equal 0.7) without intercept and the second a model like (1) without intercept. In this case the series are different in their general dynamics (and this is denoted by a distance, measured on the AR structure, equal 0.7), but equal in their GARCH structure (the distance measured on the GARCH structure is 0). In other terms, the same metrics is used for different purposes and provides different information; applying it on the structure of $y_{i,t}$ we derive information about the differences in the dynamics of two time series; applying it on the structure of $\varepsilon^2_{i,t}$, we obtain information about the differences in the dynamics of the heteroskedastic variances of two time series.

2.1. An investigation about the GARCH(1,1) distance

In this subsection we study more in detail the behavior of the distance (6), for various combination of the coefficients α_i and β_i. The behavior of the distance is clear when we pose $\beta_i = 0$ for $i = 1, 2$, which is the case of two ARCH(1) models. In this case, the distance is the difference between the two α_i coefficients. When two GARCH(1,1) models are considered, the behavior is well different; in fact, for the contemporaneous presence of α_i and β_i, similar processes can seem different. In Figure 2 the behavior of the distance between each of three particular GARCH(1,1) models and each of the 81 GARCH(1,1) models obtained varying both the coefficients in [0.1,0.9] with steps of 0.1 are shown; in the case of the models with coefficients ($\alpha = 0.1$, $\beta = 0.1$) (picture a) and ($\alpha = 0.5$, $\beta = 0.5$) (picture b), the distance has a similar behavior for the various values of α, varying β; we can note that it increases rapidly with α. In addition in

[1]In the simulations the same number of observations of the real time series used in section 3 is adopted.

Figure 2 (a) there are many initial values of the distance approximately equal zero.

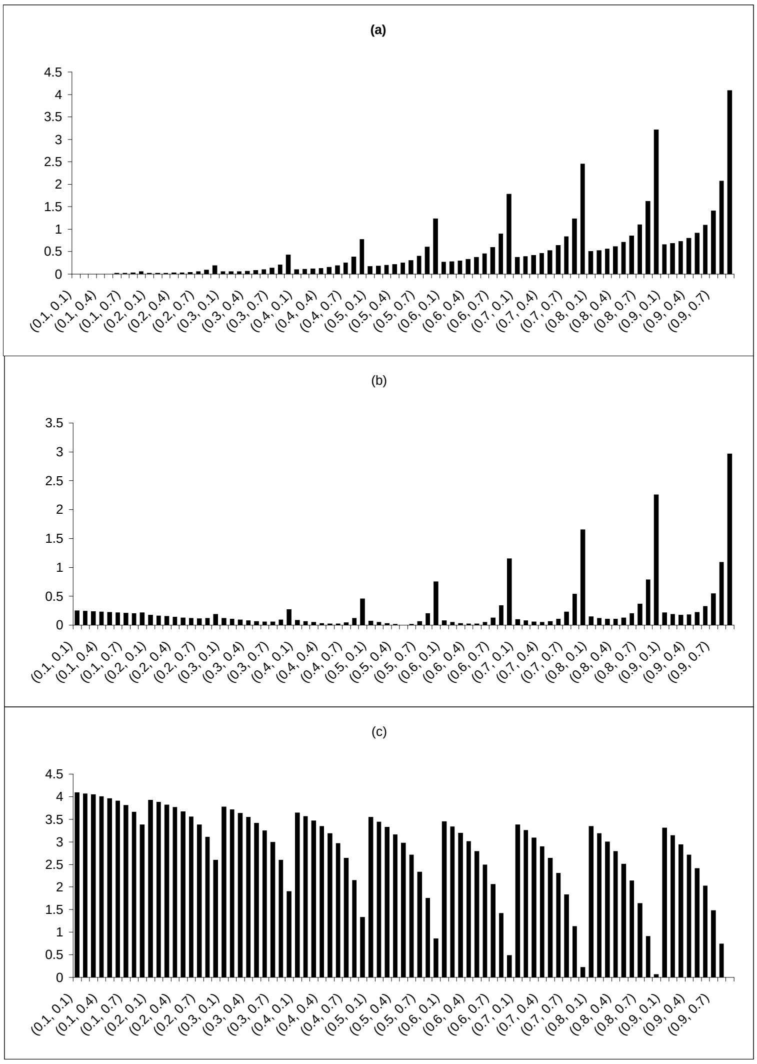

Figure 2. Distances between a fixed GARCH(1,1) model with parameters ($\alpha = 0.1$, $\beta = 0.1$) (picture a), ($\alpha = 0.5$, $\beta = 0.5$) (picture b), ($\alpha = 0.9$, $\beta = 0.9$) (picture c) and each of 81 different GARCH(1,1) models with parameters (α, β) indicated on the x axis.

Table 1. Intervals of β_2 values corresponding to 81 combinations of α_1 and β_1 for which the GARCH distance is not significantly different from zero.

	β_1								
$\alpha_1 = \alpha_2$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.1 - 0.7	0.1 - 0.7	0.1 - 0.7	0.1 - 0.7	0.1 - 0.8	0.1 - 0.8	0.3 - 0.8	0.6 - 0.8	0.9
0.2	0.1 - 0.4	0.1 - 0.5	0.1 - 0.5	0.1 - 0.6	0.2 - 0.6	0.4 - 0.7	0.6 - 0.7	0.8	0.9
0.3	0.1 - 0.3	0.1 - 0.4	0.1 - 0.5	0.2 - 0.5	0.4 - 0.6	0.5 - 0.6	0.7	0.8	0.9
0.4	0.1 - 0.3	0.1 - 0.3	0.2 - 0.4	0.3 - 0.5	0.4 - 0.5	0.6	0.7	0.8	0.9
0.5	0.1 - 0.2	0.1 - 0.3	0.2 - 0.4	0.3 - 0.5	0.5	0.6	0.7	0.8	0.9
0.6	0.1 - 0.2	0.1 - 0.3	0.2 - 0.4	0.4	0.5	0.6	0.7	0.8	0.9
0.7	0.1 - 0.2	0.1 - 0.3	0.2 - 0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.8	0.1 - 0.2	0.1 - 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.9	0.1 - 0.2	0.1 - 0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

In the case of the model with coefficients ($\alpha = 0.9$, $\beta = 0.9$) (picture

c) the distance shows a more persistent behavior. In all the cases, the equality of α_1 and α_2 or β_1 and β_2 cuts down the distance considerably.

Considering all the combination of coefficients we have the confirmation that the value of the coefficient α has a fundamental role in the calculus of the distance respect to the coefficient β. In fact, the distance is not significantly different from zero when α_1 and α_2 are different, whereas there are intervals of the β coefficient, for $\alpha_1 = \alpha_2$, for which the distance is not significantly different from zero.[2] This result is consistent with Corduas (1996), where the behavior of the distance for the ARMA case is studied. In Table 1 these intervals are shown in correspondence of all the possible combinations of α_1 and β_1, for which are calculated the critical values of the d^2 statistic (see Corduas, 1996, and the final Appendix). We can note that there are large intervals of β_2 providing distance equal 0, corresponding to small values of α_1, whereas these intervals are progressively reduced when α_1 increases. For example, the cell corresponding to $\alpha_1 = \alpha_2 = 0.3$ and $\beta_1 = 0.4$ shows that all the model with $\alpha_2 = 0.3$ and $0.2 \leq \beta_2 \leq 0.5$ have a distance not significantly different from zero respect to the model ($\alpha_1 = 0.3, \beta_1 = 0.4$). When $\alpha_1 = 0.3$ and $\beta_1 = 0.7$ the only case with distance 0 is the trivial case $\alpha_2 = 0.3$ and $\beta_2 = 0.7$.

3. Clustering the returns: an agglomerative algorithm

How could the distance developed in the previous section be used in practical cases? The most obvious application is to create homogeneous groups having a similar volatility structure. For this purpose an usual agglomerative algorithm for cluster analyses could be used; it can be developed in the following steps:

1. choose an initial benchmark series;
2. insert in the group of the benchmark series all the series with a distance from it not significantly different from zero;

[2]The test used depends on the coefficients of the GARCH models and the number of observations (we use $T = 1352$); it is described in the final appendix.

3. select the series with the minimum distance from the benchmark significantly different from zero; this series will be the new benchmark;

4. insert in the second group all the remaining series with a distance from the new benchmark not significantly different from zero;

5. repeat steps 3 and 4 until no series remain.

Table 2. GARCH(1,1) estimation (standard errors in parentheses).

	γ	α	β
cac	0.0003 (0.0001)	0.0540 (0.0081)	0.9335 (0.0094)
nik	0.0009 (0.0003)	0.0637 (0.0094)	0.9191 (0.0124)
dax	0.0005 (0.0001)	0.0965 (0.0129)	0.8876 (0.0139)
smi	0.0006 (0.0001)	0.0891 (0.0147)	0.8775 (0.0198)
fts	0.0001 (3.86E-5)	0.0443 (0.0076)	0.9492 (0.0078)
ibe	0.0006 (0.0001)	0.0838 (0.0109)	0.8920 (0.0128)
dj	0.0005 (0.0001)	0.0829 (0.0089)	0.8873 (0.0135)
bel	0.0003 (8.91E-5)	0.0989 (0.0130)	0.8863 (0.0138)
mib	0.0010 (0.0002)	0.1159 (0.0179)	0.8384 (0.0226)

Note that, differently from the common cluster algorithms, in this case the number of groups is not fixed a priori or chosen after the clustering, but it derives automatically from the algorithm.

Clearly, to classify the series we need a starting point, in the sense that the result will be different, changing the series adopted as initial benchmark. Alternatively, in applications with a small number of series, we can use each series as initial benchmark in different classifications and then verify if there are "strongest" structures.

In order to explain this algorithm, we consider the series of the returns of nine stock exchange indices from December 1, 1995 to February 5, 2001 (daily data, $T = 1352$); they refer to the following indices: CAC40 (*cac*, France), NIKKEI300 (*nik*, Japan), DAX30 (*dax*, Germany), SMI (*smi*, Switzerland), FTSE100 (*fts*, England), IBEX35I (*ibe*, Spain), DOW JONES (*dj*, U.S.A.), BEL20 (*bel*, Belgium), MIB30 (*mib*, Italy). First, a GARCH(1,1) model is estimated for each series, then the matrix of distances for each couple of series is calculated and finally the statistical test to verify the null of zero distance is applied.

The estimations of coefficients are shown in Table 2, whereas the symmetric matrix of distances in Table 3.

Table 3. Matrix of the d distances.

	cac	nik	dax	smi	fts	ibe	dj	bel	mib
cac	0.00								
nik	0.02	0.00							
dax	0.08	0.06	0.00						
smi	0.06	0.05	0.03	0.00					
fts	0.02	0.04	0.10	0.08	0.00				
ibe	0.05	0.04	0.02	0.01	0.07	0.00			
dj	0.05	0.04	0.03	0.01	0.08	0.01	0.00		
bel	0.08	0.06	0.01	0.03	0.10	0.03	0.03	0.00	
mib	0.10	0.08	0.04	0.04	0.12	0.05	0.05	0.04	0.00

In Table 4 the results of the diagnostic test for each couple of indices are shown (A indicates the case of acceptation of the null of distance 0, whereas R indicates the case of rejection).

In our example, using each series as initial benchmark, the 9 classifications provide three possible alternative consisting of two distinct groups. For example, in Figure 3 the classification obtained using the *dj* index as benchmark is shown. In this case, six indices belong to the group of *dj*; *cac* is the index with minimum distance from *dj*, significantly different from zero and *fts* has a non significant distance from *cac*.

Table 4. Test results.

	cac	nik	dax	smi	fts	ibe	dj	bel	mib
cac									
nik	A								
dax	R	R							
smi	R	A	A						
fts	A	A	R	R					
ibe	R	A	A	A	R				
dj	R	A	A	A	R	A			
bel	R	R	A	A	R	A	A		
mib	R	R	A	A	R	A	A	A	

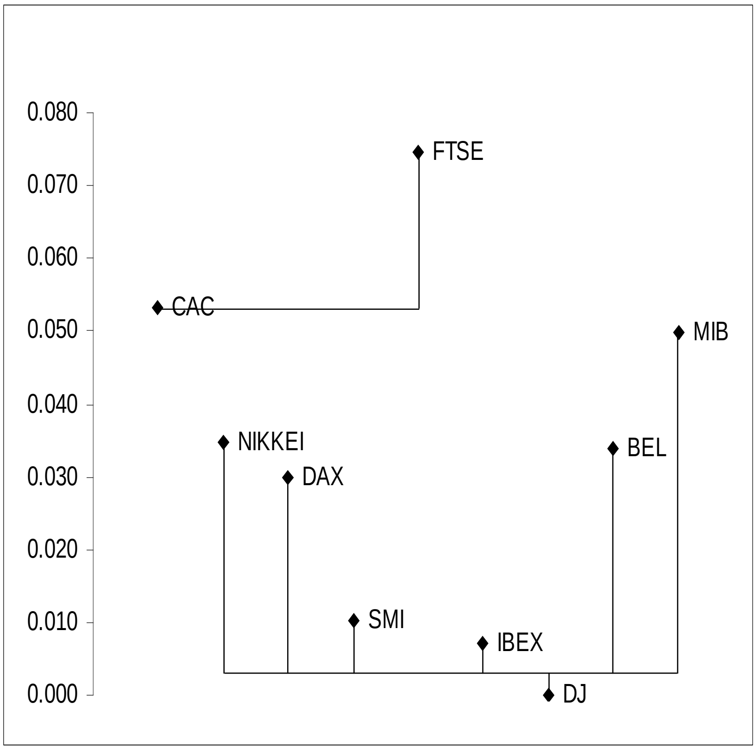

Figure 3. Tree obtained using dj as benchmark. The distances are indicated on the y axis

Using as initial benchmark *cac*, *dax*, *fts*, *bel* and *mib*, the 2 groups obtained are formed by (*cac*, *nik*, *fts*) and (*dax*, *smi*, *ibe*, *dj*, *bel*, *mib*); using as initial benchmark *smi*, *ibe* and *dj*, the 2 groups are formed by (*cac*, *fts*) and (*nik*, *dax*, *smi*, *ibe*, *dj*, *bel*, *mib*); using *nik* as initial benchmark we separate (*dax*, *bel*, *mib*) from (*cac*, *nik*, *smi*, *fts*, *ibe*, *dj*). Combining the results we deduce that there are 2 strong groups, constituted by *cac* and *fts* on a hand and *dj*, *dax*, *smi* and *ibe* on the other hand. The *nik* stays in the middle, whereas *bel* and *mib* are very similar to the *dj* group, but distant from *nik*.

These results are consistent with recent studies about the interdependence among financial markets (see, for example, Forbes and Rigobon, 2002), which show that the volatility of the Japanese market is quite autonomous respect to the other markets; for example there is not empirical evidence that the East Asian crises of October 1997 was absorbed from this market, whereas it was quickly transmitted to many European markets. The fact that the Dow Jones index is an attractor for the other markets, in terms of volatility structures, is a well known idea, having the U.S.A. economy a leading role in the global economy, and it is confirmed in these results. Concerning the European markets, it arises that the French and the English indices follow an autonomous dynamics respect to the Dow Jones.

4. Concluding remarks

In this paper an extension of the distance measure used to compare couples of ARMA models, developed by Piccolo (1984, 1989, 1990), is extended to the GARCH case. This extension provides the possibility to group the financial series having a similar volatility structure and an agglomerative algorithm was developed to obtain homogeneous clusters.

It is interesting to note that the AR metrics, born to compare the forecasting profiles of two series, seems to be particularly convenient to compare the volatility structures. This is due to the fact that the GARCH models represent the structure of the volatility, so that this metrics can capture the similarities or dissimilarities in their behavior. This fact confirms the flexibility of the AR metrics, used also for other purposes; for example, Corduas and Piccolo (1995) have used this tool to study demographic phenomena; Otranto and Triacca (2002) to develop a decision rule in the choice between direct and indirect method in seasonal adjustment. Anyway, for the GARCH case, this metrics provides a different kind of information respect to the classical AR metrics, being possible that series with different ARMA structures for the levels could have similar GARCH structures for the variances and vice versa.

The final results of the algorithm depend on the series adopted as benchmark; anyway, this is not necessarily a weak point, because generally the behavior of the markets are evaluated respect to a "dominant" market (for example, the U.S. stock exchange market, which influences the other markets or shares); on the other side, the detection of various clusters, obtained using as benchmark each market iteratively, will conduce probably to some "strong" form, or some interpretable behavior, as in the application of the previous section.

Clearly, the case of clustering is just a possible application of this instrument; another purpose could be to forecast assets, shares or stock exchange indices of the financial markets. The volatility transmission mechanisms is another field of application. In fact, the information deriving from a market can influence the behavior of another market; using the distance measure, it is possible to detect the most similar volatility structure for a certain series among a set of leading series, so that the knowledge of the latter could be used to forecast the volatility structure of the former.

Appendix: the AR metrics

In this appendix there is a brief description of the AR metrics introduced by Piccolo (1990) and the considerations above its distribution developed in Corduas (1996) with extensions to the GARCH(1,1) case.

Let V_t be a zero-mean ARMA invertible process; then, it exists a sequence $\{\pi_j\}$ such that

$$\sum_{j=1}^{\infty} |\pi_j| < \infty$$

and

$$V_t = \sum_{j=1}^{\infty} \pi_j V_{t-j} + \varepsilon_t, \tag{9}$$

where ε_t is a white noise process with variance σ^2.

Piccolo (1990) defines the distance between two ARMA invertible and independent processes V_{1t} and V_{2t} as

$$d = \left[\sum_{j=1}^{\infty} (\pi_{1j} - \pi_{2j})^2 \right]^{1/2}. \tag{10}$$

From (10) we have derived the GARCH(1,1) distance (6).

Piccolo (1989) shows that the asymptotic distribution of d^2, given the independence hypothesis, is a linear combination of independent Chi-Square variables. In order to deal with the distance measure as a test procedure, Corduas (1996) proposes to approximate the distribution of d^2 with a single Chi-Square random variable. Under the null hypothesis:

$$H_0 : \pi_{1j} = \pi_{2j} \quad \forall\, j = 1, 2, ..., \tag{11}$$

this distribution can be approximated with $a\chi^2_c + b$, where χ^2_c is a chi-squared random variable with c degree of freedom and, setting $t_i = trace\left(\widetilde{\mathbf{\Sigma}}^i\right)$:

$$a = t_3/t_2, \qquad b = t_1 - t_2^2/t_3, \qquad c = t_2^3/t_3^2. \tag{12}$$

This approximation has a good performance, as showed in Corduas (1996). In this case $\widetilde{\mathbf{\Sigma}} = \widetilde{\mathbf{\Sigma}}_1 + \widetilde{\mathbf{\Sigma}}_2$ and represents the covariance matrix of the AR coefficients in (10) under the null hypothesis. $\widetilde{\mathbf{\Sigma}}_1$ and $\widetilde{\mathbf{\Sigma}}_2$ represent respectively the estimated covariance matrices of the coefficients $\widetilde{\pi}_1 = \{\pi_{1j}\}$ and $\widetilde{\pi}_2 = \{\pi_{2j}\}$, obtained as functions of the maximum likelihood estimators of the parameters of the GARCH models, as showed in (8). For practical purposes, the vectors $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ will contain only the first k autoregressive coefficients of the representation (9), with k suitably high (for example $k = 100$).[3] The covariances matrices $\widetilde{\mathbf{\Sigma}}_1$ and $\widetilde{\mathbf{\Sigma}}_2$ can be obtained by:

$$\widetilde{\mathbf{\Sigma}}_i = \mathbf{\Gamma}_i \widetilde{\mathbf{V}}_i \mathbf{\Gamma}_i',$$

where $\widetilde{\mathbf{V}}_i$ is the covariance matrix of the estimated GARCH coefficients and $\mathbf{\Gamma}_i$ is a matrix containing the derivatives of the functions π_{ij} respect to the GARCH coefficients. For example, for the case of GARCH(1,1) model, the estimated parameters modelizing the volatility structure will be $\left(\widetilde{\alpha}_i, \widetilde{\beta}_i\right)'$, whereas $\widetilde{\pi}_i = \left(\widetilde{\alpha}_i, \widetilde{\alpha}_i\widetilde{\beta}_i, ..., \widetilde{\alpha}_i\widetilde{\beta}_i^{k-1}\right)$.

Note that, to map out Table 1, we have not performed estimation procedures, having used the theoretical covariance matrix of ARMA(1,1) processes (Brockwell and Davis, 1996). For an ARMA(1,1) process with AR coefficient equal ϕ and MA coefficient equal θ, the covariance matrix is expressed by:

$$\mathbf{V}_{ARMA} = \frac{1+\phi\theta}{T\left(\phi+\theta^2\right)} \begin{bmatrix} \left(1-\phi^2\right)\left(1+\phi\theta\right) & -\left(1-\theta^2\right)\left(1-\phi^2\right) \\ -\left(1-\theta^2\right)\left(1-\phi^2\right) & \left(1-\theta^2\right)\left(1+\phi\theta\right) \end{bmatrix} = c \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix};$$

taking into account (7), we obtain that:

$$\begin{aligned} Var\left(\alpha_i+\beta_i\right) &= Var\left(\alpha_i\right) + Var\left(\beta_i\right) + 2Cov\left(\alpha_i,\beta_i\right) = a_{11} \\ Var\left(\beta_i\right) &= a_{22} \\ Cov\left(\alpha_i+\beta_i, -\beta_i\right) &= -Cov\left(\alpha_i,\beta_i\right) - Var\left(\beta_i\right) = a_{12} \end{aligned}$$

As a consequence:

$$\mathbf{V}_{GARCH} = c \begin{bmatrix} \left(a_{11}+a_{22}+2a_{12}\right) & -\left(a_{12}+a_{22}\right) \\ -\left(a_{12}+a_{22}\right) & a_{22} \end{bmatrix}.$$

[3] For the GARCH(1,1) model it is possible to apply (6).

In this way we can apply the test considering hypothetical GARCH(1,1) processes, without estimation step; the only sample information we need is the length of the series T.

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