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Full Information Least Orthogonal Distance Estimator of structural parameters in simultaneous equation models

Alessia Naccarato
Dipartimento di Economia, Università Roma Tre
E-mail: a.naccarato@uniroma3.it

Summary: An extension of Limited Information Least Orthogonal Distance Estimator (LI LODE) for structural parameters of simultaneous equations models, to a Full Information context, is presented. The proposed extension is based on characteristic roots and vectors of a matrix deriving from the so called over-identifying restrictions.

Keywords: Simultaneous equations models, Orthogonal distance, Principal components.

1. Introduction

This paper is aimed at presenting an extension of Least Orthogonal Distance Estimator (LODE) of structural form parameters in simultaneous equation models. The LODE procedure was originally developed by Pieraccini (1988) in a Limited Information (LI) context and was based on characteristic roots and vectors of a particular variance-covariance matrix. In this article, LODE procedure is generalized to the case of full information (FI) in order to obtain simultaneous estimators for structural parameters of the complete system.

After reviewing the notation, stressing some basic ideas about the estimation problems and the identification conditions, limited information LODE is presented together with its extension to the full information context. Since the FI procedure needs the variance-covariance matrix of disturbances, a way to obtain a consistent estimator of it is presented. Finally, the consistency of the proposed estimator is shown and some concluding remarks are discussed.

2. Simultaneous equations models

Making use of standard notations, the structural form of a simultaneous equations model can be defined as follows:

$$\underset{n,m}{Y}\,\underset{m,m}{\Gamma}+\underset{n,k}{X}\,\underset{k,m}{\mathrm{B}}+\underset{n,m}{U}=\underset{n,m}{0} \tag{1}$$

where Y is the $n\times m$ matrix of endogenous variables and Γ is the corresponding $m\times m$ matrix of structural parameters, X is the $n\times k$ matrix of exogenous variables and B is the $k\times m$ matrix of their structural parameters. Finally U is the $n\times m$ matrix of disturbances for which standard hypotheses are supposed to hold:

$$\begin{aligned} &E(vecU)=0\\ &E\left(vecU(vecU)^{T}\right)=\Omega\otimes I \end{aligned} \tag{2}$$

where

$$\underset{m,m}{\Omega}=\begin{bmatrix}\sigma_1^2 & \cdots & \sigma_{1m}\\ \vdots & \ddots & \vdots\\ \sigma_{m1} & \cdots & \sigma_m^2\end{bmatrix}$$

is the variance-covariance matrix of the structural form disturbances *U*, which is supposed to be constant for all observations.

Furthermore it is assumed that:

$$\begin{aligned}
& p\lim_{n\to\infty}\frac{1}{n}U^T U = \Omega \\
& p\lim_{n\to\infty}\frac{1}{n}X^T U = \underset{k,m}{0} \\
& p\lim_{n\to\infty}\frac{1}{n}X^T X = \underset{k,k}{\Sigma_x}\,.
\end{aligned} \tag{3}$$

Under non singularity condition for Γ the *reduced form* of the equations is derived as:

$$\underset{n,m}{Y} = \underset{n,k}{X}\,\underset{k,m}{\Pi} + \underset{n,m}{V} \tag{4}$$

where:

$$\begin{aligned}
& \underset{k,m}{\Pi} = -\underset{k,m}{\mathrm{B}}\,\underset{m,m}{\Gamma^{-1}} \\
& \underset{n,m}{V} = -\underset{n,m}{U}\,\underset{m,m}{\Gamma^{-1}}\,.
\end{aligned} \tag{5}$$

The last equation in (5) represents the matrix of disturbances of reduced form, for which it is possible to write:

$$\begin{aligned}
& E(V) = 0 \\
& E\left(V^T V\right) = n\left(\Gamma^{-1}\right)^T \Omega\,\Gamma^{-1}.
\end{aligned} \tag{6}$$

Post-multiplying by Γ the first equation in (5) we obtain:

$$\underset{k,m}{\Pi}\,\underset{m,m}{\Gamma} = -\underset{k,m}{\mathrm{B}} \tag{7}$$

which represents the relation between reduced and structural form parameters.

Since (7) is a system of k equations with $m \times (m+k)$ unknowns, exclusion constrains are introduced in order to find the solution with respect to Γ and B in terms of Π.

If – as it usually happens – each equation does not include all the endogenous and exogenous variables, it is possible to consider the following partition of the overall matrix of endogenous variables with respect to *i-th* structural form equation:

$$\underset{n,m}{Y} = \left[\underset{n,m_{1i}}{Y_{1i}} \; \vdots \; \underset{n,m_{2i}}{Y_{2i}} \right]$$

where the first m_{1i} columns refer to the endogenous variables included in *i-th* equation and the last m_{2i} columns refer to those excluded. In the same way the vectors of Γ's in *i-th* equation can be reordered as:

$$\underset{m,1}{\Gamma_i} = \begin{bmatrix} \underset{m_{1i},1}{\Gamma_{1i}} \\ \cdots \\ \underset{m_{2i},1}{0} \end{bmatrix}$$

where the first m_{1i} elements of Γ_i refer to endogenous variables included in the *i-th* equation. Notice that defining the vector Γ_i no normalization rule has yet been introduced.

Similarly, let us consider the partition:

$$\underset{n,k}{X} = \left[\underset{n,k_{1i}}{X_{1i}} \; \vdots \; \underset{n,k_{2i}}{X_{2i}} \right]$$

where X_{1i} and X_{2i} are the sub-matrices corresponding to the exogenous variables, included in and excluded from, the *i-th* equation. Accordingly let us define

$$\underset{k,1}{\mathrm{B}_i} = \begin{bmatrix} \underset{k_{1i},1}{\mathrm{B}_{1i}} \\ \cdots \\ \underset{k_{2i},1}{0} \end{bmatrix}$$

where the first k_{1i} parameters are related to the exogenous variables included in the *i-th* equation.

Therefore the *i-th* structural equation can be expressed as:

$$Y_{1i}\Gamma_{1i} + X_{1i}\mathrm{B}_{1i} + U_i = 0 \,.$$

According to each equation of the system a different ordering of variables has to be performed.

3. Condition for identification

Usually rank conditions for identification of a simultaneous equation system, as well as order conditions, are obtained after applying the normalization rule: in our case, this doesn't happen so that we have to redefine the identifiability condition.

With respect to the *i-th* structural equation of the system, relation (7) can be written as:

$$\begin{cases} \underset{k_{1i},m_{1i}}{\Pi^i_{11}} \; \underset{m_{1i},1}{\Gamma_{1i}} = -\underset{k_{1i},1}{\mathrm{B}_{1i}} \\ \underset{k_{2i},m_{1i}}{\Pi^i_{12}} \; \underset{m_{1i},1}{\Gamma_{1i}} = \underset{k_{2i},1}{0} \,. \end{cases} \tag{8a}$$

or

$$\begin{bmatrix} \underset{k_{1i},m_{1i}}{\Pi^i_{11}} & I_{k_{1i}} \\ \underset{k_{2i},m_{1i}}{\Pi^i_{12}} & \underset{k_{2i},k_{1i}}{0} \end{bmatrix} \begin{bmatrix} \underset{m_{1i},1}{\Gamma_{1i}} \\ \underset{k_{1i},1}{\mathrm{B}_{1i}} \end{bmatrix} = 0 \tag{8}$$

where Π^i_{11} refers to the *i-th* equation RF parameters of endogenous and exogenous variables included, while Π^i_{12} refers to the endogenous included and exogenous excluded ones.

Defining the matrix Π_*^i in the following way:

$$\underset{k,\,m_{1i}+k_{1i}}{\Pi_*^i} = \begin{bmatrix} \underset{k_{1i},\,m_{1i}}{\Pi_{11}^i} & I_{k_{1i}} \\ \underset{k_{2i},\,m_{1i}}{\Pi_{12}^i} & \underset{k_{2i},\,k_{1i}}{0} \end{bmatrix},$$

the rank condition for solving the system (8a) takes the following form.

Condition 1 – System (8a) admits a unique solution – up to a proportionality constant – if the rank:

$$r\left(\Pi_*^i\right) = m_{1i} + k_{1i} - 1 \tag{9}$$

the proof follows directly from the Rouchè-Capelli theorem.

Condition 2 – $r\left(\Pi_*^i\right) = m_{1i} + k_{1i} - 1$ if and only if

$$r\left(\Pi_{12}^i\right) = m_{1i} - 1\,. \tag{10}$$

Proof – Let us consider the four blocks partitioned matrix Π_*^i:

$$\Pi_*^i = \begin{bmatrix} \Pi_{11}^i & I_{k_{1i}} \\ \Pi_{12}^i & 0 \end{bmatrix}.$$

Remembering that:

$$r\left(\Pi_*^i\right) \leq r\left(\Pi_{11}^i\,,\ I_{k_{1i}}\right) + r\left(\Pi_{12}^i\,,\ 0\right),$$

and noticing that the first block including the identity matrix has always full rank:

$$r\left(\Pi_{11}^i,\ I_{k_{1i}}\right) = k_{1i}$$

then condition (9) holds if and only if

$$r\left(\Pi_{12}^{i}\right) = m_{1i} - 1 .$$

If equation (10) holds the system (8a) admits a unique non trivial solution up to a proportionality constant. In this case the structural parameters are said to be identified.

If

$$r\left(\Pi_{*}^{i}\right) < m_{1i} + k_{1i} - 1, \tag{11}$$

the structural parameters are said to be under-identified and system (8a) admits no solution.

When the reduced form parameters Π are substituted with their OLS estimates $\hat{\Pi}$, the system (8a) becomes:

$$\begin{cases} \underset{k_{1i},m_{1i}}{\hat{\Pi}_{11}^{i}} \underset{m_{1i},1}{\Gamma_{1i}} + \underset{k_{1i},1}{\mathrm{B}_{1i}} = \underset{k_{1i},1}{\varepsilon_{1i}} \\ \underset{k_{2i},m_{1i}}{\hat{\Pi}_{12}^{i}} \underset{m_{1i},1}{\Gamma_{1i}} = \underset{k_{2i},1}{\varepsilon_{2i}} \end{cases} \tag{12}$$

so that in both equations an error component occurs because of the use of the estimates $\hat{\Pi}$ instead of the true Π values: then rank conditions cannot be verified. The rank of $\hat{\Pi}_{12}^{i}$ cannot therefore be used as an identification criterion and we need to define the so-called "order conditions" which are related to the number of the equations and unknowns in the system (8a) and are a direct consequence of rank conditions.

Condition 3 – If rank condition (9) is satisfied, the matrix $\hat{\Pi}_{*}^{i}$ has to be of order greater or equal to $m_{1i} + k_{1i} - 1$, that is:

$$k \geq k_{1i} + m_{1i} - 1$$

i.e.:

$$k_{2i} \geq m_{1i} - 1$$

and the number of excluded exogenous variables has to be greater than or equal to the number of included endogenous variables minus one, which is the formulation generally used for order conditions.

Exact identification will occurs when:

$$k_{1i} = m_{1i} - 1,$$

while under identification occurs when:

$$k_{1i} < m_{1i} - 1.$$

In the first case there is a unique solution while in the second one there is no solution.

4. Limited information and full information LODE

LODE estimator is – in its original formulation – a limited information method, i.e. an estimator of structural parameters, equation by equation (Pieraccini, 1988). Since it is well known that FI estimators are asymptotically more efficient than LI (Goldberger, 1964, pp. 346-356, Judge *et al.*, 1985), it is worthwhile to generalize LODE method to a full information context.

Let's now first consider the case of LI LODE.

Defining:

$$\underset{k,m_{1i}+k_{1i}}{\hat{\Pi}_*^i} = \begin{bmatrix} \underset{k_{1i},m_{1i}}{\hat{\Pi}_{1i}} & I_{k_{1i}} \\ \underset{k_{2i},m_{1i}}{\hat{\Pi}_{2i}} & \underset{k_{2i},k_{1i}}{0} \end{bmatrix}, \quad \underset{m_{1i}+k_{1i},1}{\delta_i} = \begin{bmatrix} \underset{m_{1i},1}{\Gamma_{1i}} \\ \underset{k_{1i},1}{\mathrm{B}_{1i}} \end{bmatrix}, \quad \underset{k,1}{\varepsilon_i} = \begin{bmatrix} \underset{k_{1i},1}{\varepsilon_{1i}} \\ \underset{k_{2i},1}{\varepsilon_{2i}} \end{bmatrix},$$

equation (12) can be written as:

$$\hat{\Pi}^i_* \delta_i = \varepsilon_i \tag{13}$$

where it can be shown that:

$$\underset{k,1}{\varepsilon_i} = \left(X^T X\right)^{-1} X^T U_i . \tag{14}$$

To have uncorrelated residuals it is possible to make a transformation which leads to the following expression of the variance of *i-th* equation disturbances:

$$\delta_i^T \hat{\Pi}^{i\,T}_* \left(X^T X\right) \hat{\Pi}^i_* \delta_i \tag{15}$$

Limited information LODE is given by the vector δ_i which minimizes the quadratic form (15) and it is then given by the eigenvector associated to the smallest eigenvalue of the matrix $\hat{\Pi}^{i\,T}_* \left(X^T X\right) \hat{\Pi}^i_*$.

If we make the following position:

$$\begin{bmatrix} \hat{\Pi}^i_{11} \\ \hat{\Pi}^i_{12} \end{bmatrix} = \hat{\Pi}^i_1 = \left(X^T X\right)^{-1} X^T Y_{1i} ,$$

it can be easily shown that (15) reduces to:

$$\underset{m_{1i}+k_{1i},k}{\hat{\Pi}^{i\,T}_*} \underset{k,k}{\left(X^T X\right)} \underset{k,m_{1i}+k_{1i}}{\hat{\Pi}^i_*} = \begin{bmatrix} \underset{m_{1i},m_{1i}}{\hat{\Pi}^{i\,T}_1 X^T X \hat{\Pi}^i_1} & \underset{m_{1i},k_{1i}}{\hat{\Pi}^{i\,T}_1 X^T X_{1i}} \\ \underset{k_{1i},m_{1i}}{X^T_{1i} X \hat{\Pi}^i_1} & \underset{k_{1i},k_{1i}}{X^T_{1i} X_{1i}} \end{bmatrix} =$$

$$= \begin{bmatrix} \underset{m_{1i},m_{1i}}{Y^T_{1i} X\left(X^T X\right)^{-1} X^T Y_{1i}} & \underset{m_{1i},k_{1i}}{Y^T_{1i} X_{1i}} \\ \underset{k_{1i},m_{1i}}{X^T_{1i} Y_{1i}} & \underset{k_{1i},k_{1i}}{X^T_{1i} X_{1i}} \end{bmatrix} = \underset{m_{1i}+k_{1i},m_{1i}+k_{1i}}{A_{1i}} \tag{15a}$$

where the meaning of the symbol A_{ii} will become clear in few lines.

Then (15) becomes:

$$\delta_i^T A_{ii} \delta_i \tag{16}$$

so that LODE estimator $\hat{\delta}_i$ is defined in terms of the eigenvalues and eigenvectors of matrix A_{ii}.

Let us now consider the extension of LODE method to FI context.

Relations between reduced and structural form parameters for the whole system of equation are given by:

$$\begin{bmatrix} \underset{k,m_{11}+k_{11}}{\hat{\Pi}_*^1} & \underset{k,m_{12}+k_{12}}{0} & \cdots & \underset{k,m_{1m}+k_{1m}}{0} \\ \underset{k,m_{11}+k_{11}}{0} & \underset{k,m_{12}+k_{12}}{\hat{\Pi}_*^2} & \cdots & \underset{k,m_{1m}+k_{1m}}{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underset{k,m_{11}+k_{11}}{0} & \underset{k,m_{12}+k_{12}}{0} & \cdots & \underset{k,m_{1m}+k_{1m}}{\hat{\Pi}_*^m} \end{bmatrix} \begin{bmatrix} \underset{m_{11}+k_{11},1}{\delta_1} \\ \underset{m_{12}+k_{12},1}{\delta_2} \\ \vdots \\ \underset{m_{1m}+k_{1m},1}{\delta_1} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{bmatrix} \tag{17}$$

or in a more compact form, using a self evident notation:

$$\underset{mk,s}{\hat{\Pi}_*} \underset{s,1}{\delta} = \underset{mk,1}{\varepsilon} \tag{17a}$$

where the number s of colums of matrix $\hat{\Pi}_*$ is defined as

$$s = \sum_{i=1}^{m} \left(m_{1i} + k_{1i} \right).$$

From equation (14) applied to the vector ε, the variance-covariance matrix of the error component is:

$$E\left(\varepsilon\varepsilon^T\right) = \underset{mk,mk}{\Sigma} = \underset{m,m}{\Omega} \otimes \left(\underset{k,k}{X^T X} \right)^{-1}. \tag{18}$$

As in the case of LI to obtain uncorrelated errors it is possible to make a transformation leading to the following expression:

$$W^T W = \underset{1,S}{\delta^T} \underset{S,mk}{\hat{\Pi}_*^T} \underset{mk,mk}{\left(\Omega \otimes \left(X^T X\right)^{-1}\right)^{-1}} \underset{mk,S}{\hat{\Pi}_*} \underset{S,1}{\delta} = \delta^T \hat{\Pi}_*^T \left(\Omega^{-1} \otimes \left(X^T X\right)\right) \hat{\Pi}_* \delta \qquad (19)$$

which represents the trace of Ω matrix.

Full Information LODE is obtained by minimizing the quadratic form (19) i. e. by considering the eigenvector associated with the smallest eigenvalue of the matrix:

$$\underset{S,S}{A} = \hat{\Pi}_*^T \left(\Omega^{-1} \otimes \left(X^T X\right)\right) \hat{\Pi}_* \qquad (20)$$

To find the vector δ which minimizes (19) is then equivalent to minimize the trace of Ω matrix.
The explicit form of A matrix is:

$$A = \begin{bmatrix} \hat{\Pi}_*^{1\,T} \sigma^{11} \left(X^T X\right) \hat{\Pi}_*^1 & \cdots & \hat{\Pi}_*^{1\,T} \sigma^{1i} \left(X^T X\right) \hat{\Pi}_*^i & \cdots & \hat{\Pi}_*^{1\,T} \sigma^{1m} \left(X^T X\right) \hat{\Pi}_*^m \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \hat{\Pi}_*^{i\,T} \sigma^{i1} \left(X^T X\right) \hat{\Pi}_*^1 & \cdots & \hat{\Pi}_*^{i\,T} \sigma^{ii} \left(X^T X\right) \hat{\Pi}_*^i & \cdots & \hat{\Pi}_*^{i\,T} \sigma^{im} \left(X^T X\right) \hat{\Pi}_*^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\Pi}_*^{m\,T} \sigma^{m1} \left(X^T X\right) \hat{\Pi}_*^1 & \cdots & \hat{\Pi}_*^{m\,T} \sigma^{mi} \left(X^T X\right) \hat{\Pi}_*^i & \cdots & \hat{\Pi}_*^{m\,T} \sigma^{mm} \left(X^T X\right) \hat{\Pi}_*^m \end{bmatrix}$$

where σ^{ij} are the element of the matrix Ω^{-1}.

The block-diagonal elements of $\underset{S,S}{A}$ are of the form (15a) – now it is clear the reason for using the proposed notation – whereas the extra-diagonal block elements are:

$$A_{ij} = \sigma^{ij} \begin{bmatrix} \underset{m_{1i},m_{1j}}{\hat{\Pi}_1^{i\,T} X^T X \hat{\Pi}_1^j} & \underset{m_{1i},k_{1j}}{\hat{\Pi}_1^{i\,T} X^T X_{1j}} \\ \underset{m_{1i},m_{1j}}{X_{1i}^T X \hat{\Pi}_1^{j\,T}} & \underset{m_{1i},k_{1j}}{X_{1i}^T X_{1j}} \end{bmatrix}. \qquad (21)$$

Taking into account the definition of $\hat{\Pi}_*^i$ and $\hat{\Pi}_*^j$ the extra-diagonal block elements of $\underset{S,S}{A}$ can be written as in (15a):

$$A_{ij} = \sigma^{ij} \begin{bmatrix} \underset{m_{1i},m_{1j}}{Y_{1i}^T X \left(X^T X\right)^{-1} X^T Y_{1j}} & \underset{m_{1i},k_{1j}}{Y_{1i}^T X_{1j}} \\ \underset{k_{1i},m_{1j}}{X_{1i}^T Y_{1j}} & \underset{k_{1i},k_{1j}}{X_{1i}^T X_{1j}} \end{bmatrix}. \tag{22}$$

The eigenvector associated with the smallest eigenvalue of matrix $\underset{S,S}{A}$ will then minimize the quadratic form (19), i.e. the trace of the variance-covariance matrix of structural form disturbances.

Full Information LODE is obtained multiplying the eigenvector associated to the smallest eigenvalue of matrix A through m constants defined as the reciprocal of the elements corresponding to the endogenous variables at right hand sides in each SF equation.

If a is the smallest eigenvalue of $\underset{S,S}{A}$ and P_a is the associated eigenvector, then:

$$\hat{\delta} = C\, P_a \tag{23}$$

where C is the block diagonal matrix defined as follows:

$$C = \begin{bmatrix} c_1 I_{m_1+k_1} & & \\ & \ddots & \\ & & c_m I_{m_1+k_1} \end{bmatrix} \tag{24}$$

and

$$c_i = -\frac{1}{p_{0i}} \tag{25}$$

with p_{0i} being the eigenvector's element corresponding to the endogenous variable y_{oi} at left hand side in *i-th* structural form equation.

It has to be noticed that FI LODE could have computational advantages with respect to FIML which, in non standard problems, converges slowly to solutions or may achieve a local maximum instead of the absolute one.

5. Estimation of variance-covariance matrix Ω

Equation (19), which defines explicitly the quadratic form to be minimized, is a function of disturbances variance-covariance matrix Ω which is unknown. Then it is necessary to estimate it.

As usual it is possible to go through a two stage procedure: in the first stage to estimate the SF parameters through LI LODE and use them to compute $\hat{U}$ i. e. the matrix of disturbances of SF:

$$\hat{U} = -\hat{V}\,\hat{\Gamma}$$

where $\hat{V}$ is the matrix of residuals of OLS estimators of RF equations.

Then an estimate of the variance-covariance matrix is obtained as:

$$\hat{\Omega} = \begin{bmatrix} g_1^{-1/2} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \cdots & \cdots & \vdots \\ 0 & \cdots & g_i^{-1/2} & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & g_m^{-1/2} \end{bmatrix} \hat{U}^T \hat{U} \begin{bmatrix} g_1^{-1/2} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \cdots & \cdots & \vdots \\ 0 & \cdots & g_i^{-1/2} & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & g_m^{-1/2} \end{bmatrix}$$

where:

$$g_i = \frac{1}{n - m_{1i} - k_{1i}}.$$

In the second stage structural parameters estimates are obtained introducing $\hat{\Omega}$ in equation (19). Then Full Information LODE is proportional to the eigenvector associated to the smallest eigenvalue of :

$$\hat{A} = \hat{\Pi}_*^T \left(\hat{\Omega}^{-1} \otimes \left(X^T X\right)\right)\hat{\Pi}_*$$

from which they can be obtained as in (23).

6. Consistency of Full Information LODE

In this section we show that FI LODE consistently estimates structural form parameters. This results generalizes a previous result derived for LI LODE by Perna (1988).

To this end, let us assume conditions (2) and (3) and – in addition – that the exogenous variables matrix $\underset{n,k}{X}$ is non random and has full rank:

$$r\left(\underset{n,k}{X}\right) = k .$$

Under these conditions the following Lemma is true.

Lemma: The eigenvector associated to the smallest eigenvalue of the matrix $\Pi_*^T \left(\Omega^{-1} \otimes \left(X^T X\right)\right)\Pi_*$ is proportional to δ according to m constants of proportionality.

Proof: If the Π matrix is known then, it is

$$\Pi_*^T \left(\Omega^{-1} \otimes \left(X^T X\right)\right)\Pi_* \delta = 0$$

If α is the smallest eigenvalue of matrix $\Pi_*^T\left(\Omega^{-1} \otimes \left(X^T X\right)\right)\Pi_*$ then it has to be $\alpha = 0$.

If Ψ_α is the associated eigenvector, it is possible to write

$$\Psi_\alpha^T \Pi_*^T \left(\Omega^{-1} \otimes \left(X^T X\right)\right)\Pi_* \Psi_\alpha = 0 \qquad (26)$$

and setting:

$$\Pi_* \Psi_\alpha = z$$

(26) becomes:

$$z^T \left(\Omega^{-1} \otimes \left(X^T X\right)\right) z = 0 \qquad (27)$$

where the matrix $\left(\Omega^{-1} \otimes \left(X^T X\right)\right)$ is positive definite, since Ω is the variance-covariance matrix of disturbances, and the matrix X is supposed to have full rank. Then condition (27) is true if and only if $z = 0$, i. e. if and only if:

$$\Pi_* \Psi_\alpha = 0 . \qquad (28)$$

It follows that Ψ_α has to be proportional to δ - since δ is the vector of parameters for which (26) is true. Then the following relation has to be verified:

$$\underset{S,1}{\Psi_\alpha} = \underset{S,S}{\Xi^{-1}} \underset{S,1}{\delta}$$

that is:

$$\underset{S,1}{\delta} = \underset{S,S}{\Xi} \underset{S,1}{\Psi_\alpha}$$

where the matrix Ξ is:

$$\Xi = \begin{bmatrix} \xi_1 \underset{m_1 1+k_{11}}{I} & & & & \\ & \ddots & & & \\ & & \xi_i \underset{m_{i1}+k\, i_1}{I} & & \\ & & & \ddots & \\ & & & & \xi_m \underset{m_{m1}+k_{m1}}{I} \end{bmatrix}$$

The vector δ will then be:

$$\underset{S,1}{\delta} = \begin{bmatrix} \xi_1 \underset{m_1 1+k_{11}}{I} & & & & \\ & \ddots & & & \\ & & \xi_i \underset{m_{i1}+k\, i_1}{I} & & \\ & & & \ddots & \\ & & & & \xi_m \underset{m_{m1}+k_{m1}}{I} \end{bmatrix} \begin{bmatrix} \underset{m_{11}+k_{11},1}{\Psi_{\alpha 1}} \\ \vdots \\ \underset{m_{i1}+k_{i1},1}{\Psi_{\alpha i}} \\ \vdots \\ \underset{m_{m1}+k_{m1},1}{\Psi_{\alpha m}} \end{bmatrix}$$

where $\xi_i = \frac{1}{\psi_{0i}}$, and ψ_{0i} is the element corresponding to the endogenous variable at left hand side of the structural equation with respect to which the normalization rule has to be performed and $\Psi_{\alpha i}$ be the vector of Ψ_{α} corresponding to *i-th* equation.

Theorem: Full Information LODE consistently estimates structural form parameters.

Proof: It is well known that OLS estimator of RF parameters $\hat{\Pi}$ are consistent, so that:

$$\underset{n\to\infty}{p\lim}\, \hat{\Pi}_*^T \left(\Omega^{-1} \otimes X^T X\right) \hat{\Pi}_* = \Pi_*^T \left(\Omega^{-1} \otimes X^T X\right) \Pi_* \,. \qquad (29)$$

Furthermore, since the eigenvalues of a matrix are differentiable functions of its elements (Kato, 1982) it follows that:

$$p\lim_{n\to\infty} a = \alpha = 0$$

and

$$p\lim_{n\to\infty} P_a = \Psi_\alpha \tag{30}$$

Because of the previous Lemma the vector of SF parameters δ is proportional to the eigenvector Ψ_α, so that it will be

$$p\lim_{n\to\infty} \hat{\delta} = \delta \tag{31}$$

which proves the Theorem.

Because of this Theorem it follows that also the proposed estimation of Ω will give raise to consistent estimators.

7. Conclusions

In current literature there are two approaches to the study of estimator's statistical properties. The first one is aimed at finding small sample exact distribution of estimators while the other one is to compare estimator's through simulation experiments.

Several contributions belongs to the first research line (Basmann, 1961; Sargan, 1976; Kunitomo *et al.*, 1983; Morimune, 1983, 2001). However, general results have not yet been achieved, because of the very specific models that have been considered. As it is well known the main problem is the non linear relationship among structural and reduced form parameters.

In the second case, it is known that the results of an experiment are influenced by the simulation hypoteses. Modifying these assumptions and considering a large number of experiments it is possible to drow some general conclusion.

For LI LODE, simulation experiments have shown that the method has good performance with respect to other LI estimators (Perna, 1989; Cau, 1990; Sbrana, 2001; Zurlo, 2006). A complete report of this Montecarlo study will be the object of a next contribution together with the results of a simulation experiment on FI LODE.

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