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Partial and total orderings of dependence on tables with given margins

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Summary: The purpose of this paper is the study of dependence for $r \times s$ tables in the class of all bivariate tables with given margins. First of all, a dependence partial ordering will be defined, based on the signs and the absolute values of $r \times s$ contingencies. Then, to increase the number of couples of comparable tables, a new partial ordering, called intensity dependence ordering and based only on the absolute value of contingencies, is introduced. To allow the comparison of all pairs of tables, a dependence total ordering is needed, therefore a broad class of functions - able to induce total synthetic dependence orderings – is introduced, their arguments being the absolute values of relative contingencies. Among them, Mortara's and Pearson's indexes can be identified.
Finally, the dependence degree of a given table can be expressed by its relative position in a chosen dependence ordering of all the tables in the same class. This methodology is applied, as an example, to four classes of tables.

Keywords: Dependence ordering, Directional dependence ordering, Intensity dependence ordering, Synthetic dependence ordering.

1. Introduction

The relevant work of distinguished scholars, such as K.Pearson, Benini and Mortara, forged the study of dependence between two

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This note is the result of a close collaboration, even if, more specifically, the Introduction and Section 3 are due to M. Zenga, the other sections to F. Greselin.

qualitative variables, during the period from the last decades of the 19th century to the first decades of the 20th century. Benini introduced attraction and rejection indexes, for each cell in a two-way table (Benini, 1901). Mortara introduced the mean absolute contingency index (Mortara, 1922), an association measure defined as the weighted arithmetic mean of the absolute value of relative contingencies, with the independence frequencies as weights. Pearson's coefficient of contingency (Pearson, 1904) is similarly based on the weighted quadratic mean of the same quantities.

To measure dependence, Gini proposed a new approach (Gini, 1954-55), based on the following two steps: first of all evaluating all the dissimilarity indexes between each conditional distribution and the corresponding marginal distribution, and successively synthesizing these indexes by a mean. In this way he obtained the so-called *indici di connessione totale*. Later on, Castellano coined the term *indici di connessione globale*, to designate the mean value of the dissimilarity indexes between all conditional distributions (Castellano, 1960).

Since earlier association studies, the distinction concerning the pre-existence of marginal frequencies with respect to joint frequencies (Castellano, 1962; Zenga, 1964) deserved a great deal of attention. Theoretical aspects related to the hypothesis for which both marginal distributions are fixed appeared to be very attractive. Fréchet introduced the class of all bivariate tables having the same marginal data (Fréchet, 1951). Leti, by supposing that each unit of a population is a pair, developed the concept of distribution of all tables with given marginals (Leti, 1970); moreover, he proved that the independence frequency of every cell coincides with the arithmetic mean of the frequencies of the same cell in all tables of the Fréchet class. A careful analysis concerning the 'nature' of marginal data, particularly for the characterization of maximum dependence situations, was carried out in Zanella's monograph on dependence (Zanella, 1988).

Usually, scholars distinguish between two extreme situations in dependence studies: distributive independence and maximum dependence. In order to measure the degree of dependence of a table where none of these situations exists, they turn to the evaluation of standardized indexes, usually assuming values in the range [0,1] and attaining their endpoints in the two extreme cases (Bonferroni, 1941; Faleschini, 1948). While distributive independence is uniquely characterized, maximum dependence needs some further specifications: it may be complete or absolute[1] (*unilateral* or *bilateral*, respectively, in the Italian literature: see Leti, 1983; Zanella, 1988; Zenga, 1988). In the case of fixed marginal frequencies, the identification of maximum dependence is not straightforward (Zanella, 1988).

A great improvement of many topics in statistics (concentration, variability, kurtosis, etc.) came from the introduction of partial ordering relations (concentration ordering, variability ordering, etc.) between variables and from the definition of suitable transformations of variables on these partial orderings (Van Zwet, 1964). The introduction of a partial ordering relation implies that a good statistical index has to be coherent with such ordering.

The aim of this work is to propose new partial orderings in the study of dependence of two categorical variables. While for quantitative and for ordinal variables a vast assortment of bivariate dependence orderings appears in literature, only few works deal with nominal categorical variables. They will be briefly recalled in the beginning of Section 3. The definition of a suitable transfer involving two frequencies in a two-way table that can decrease (increase) dependence - without modifying marginal frequencies - is the main topic of Section 3. According to these transfers, a partial ordering relation, called 'directional dependence ordering' and denoted by $\prec_{DD}$ (DD-ordering), can be introduced among the tables with given margins (the reference class). In the DD-ordering both the sign of each contingency and its absolute value are considered. In Section 4 some numerical examples, showing how the DD-ordering works, are provided.

To increase the number of pairs of comparable tables one can weaken the relation $\prec_{DD}$, by considering only the absolute value of each

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[1] The above definitions are briefly recalled here: "Considering a population classified according to the presence or absence of two attributes A and B, we shall say that association is complete if all A's are B's. Absolute association arises when all A's are B's and all B's are A's." (Kendall and Stuart, 1979, p. 560)

contingency. Hence, in Section 5, a new partial ordering, called 'intensity dependence ordering' and denoted by $\prec_{\text{ID}}$ (ID-ordering) is defined.

To compare all pairs of tables, the definition of a total ordering relation is needed. Hence, in Section 6, the notion of 'synthetic dependence total ordering', denoted by $\prec_{\text{SD}}$, is given. Therefore a class of functions - able to induce an SD-ordering – is introduced, their arguments being the absolute values of relative contingencies. Among them, Mortara's and Pearson's indexes can be identified. The hierarchical structure of the orderings $\prec_{\text{DD}}$ $\prec_{\text{ID}}$ and $\prec_{\text{SD}}$ is then remarked. Finally, given a table *T* in the reference class, some useful information about its degree of dependence can be achieved by observing how many tables, among all those comparable with *T*, are dominated by *T* (in a partial or a total dependence ordering). In this way a meaningful measure of dependence can be defined.

2. Terminology

Let *n* statistical units of a given population be classified according to the qualitative variables *A* and *B*, both with nominal scale, with a finite number of unordered categories, denoted by $a_1,\dots,a_j,\dots,a_r$ and $b_1,\dots,b_i,\dots,b_s$ respectively. As usual, the joint frequency $n(a_i,b_j)$ of the pair of modalities a_i and b_j is denoted by n_{ij}, while marginal frequencies are denoted by $n_{i\bullet} = n(a_i)$, $i = 1,2,\dots,r$ for variable *A* and $n_{\bullet j} = n(b_j)$, $j = 1,2,\dots,s$, for variable *B*. Bivariate statistical data are usually represented in a table with *r* rows and *s* columns:

$A \backslash B$	b_1		b_j		b_s	*Total*
a_1	n_{11}	…	n_{1j}	…	n_{1s}	$n_{1\bullet}$
…	…		…		…	…
a_i	n_{i1}	…	n_{ij}	…	n_{is}	$n_{i\bullet}$
…	…		…		…	…
a_r	n_{r1}	…	n_{rj}	…	n_{rs}	$n_{r\bullet}$
Total	$n_{\bullet 1}$		$n_{\bullet j}$		$n_{\bullet s}$	n

Henceforth, let:

$$\hat{n}_{ij} = \frac{n_{i\bullet} \cdot n_{\bullet j}}{n} \qquad (1)$$

be the value of the generic joint frequency, in the hypothesis of independence.

In association studies, the contingencies:

$$c_{ij} = n_{ij} - \hat{n}_{ij} \quad (i = 1,2,..,r;\ j = 1,2,...,s) \qquad (2)$$

play a fundamental role. To evaluate deviation from independence, one can also employ the relative contingencies:

$$\rho_{ij} = \frac{n_{ij} - \hat{n}_{ij}}{\hat{n}_{ij}}. \qquad (3)$$

Three distinct possibilities, reflecting the typology of statistical data, arise in the study of dependence between variables *A* and *B* (Kendall and Stuart, 1979; Leti, 1983; Zanella, 1988):

a) both sets of marginal frequencies are fixed;
b) the marginal frequencies of one variable are considered fixed;
c) the total population size n is fixed.

In this work we refer to case a), in which both margins are fixed in advance.

Definition 1: *The reference class* (Fréchet, 1951, p. 53)

The class $\mathcal{T}(n_{i\bullet}\,;\,n_{\bullet j})$ is the collection of all $r \times s$ contingency tables, with integer entries n_{ij}, whose row sums $n_{i\bullet}$ and column totals $n_{\bullet j}$ are considered fixed.

Remarks:

i) Depending on the specific values of these marginal frequencies and the grand total $n = n_{1\bullet} + \ldots + n_{r\bullet} = n_{\bullet 1} + \ldots + n_{\bullet s}$, the independence table $T\{\hat{n}_{ij}\}$ with entries $\hat{n}_{ij}$ may not belong to $\mathcal{T}(n_{i\bullet}\,;\,n_{\bullet j})$.

ii) All the tables $T \in \mathcal{T}(n_{i\bullet}\,;\,n_{\bullet j})$ share the same independence frequencies $\hat{n}_{ij}$.

iii) $\mathcal{T}(n_{i\bullet}\,;\,n_{\bullet j})$ is a finite set of tables: to locate and single out all the tables $T\{n_{ij}\} \in \mathcal{T}(n_{i\bullet}; n_{\bullet j})$ see, among others (Leti, 1970; Greselin, 2003).

3. Frequency transfers and directional dependence

The study of partial orderings relations for contingency tables arises, in literature, in the context of majorization orderings (Marshall and Olkin, 1979): see the works of Joe, Forcina and Giovagnoli, and Scarsini, whose contributions are briefly recalled here, as an important reference.

In order to compare the relative degree of dependence of two tables T and T' in $\mathcal{T}(n_{i\bullet}; n_{\bullet j})$ Joe proposes to compare the column vectors vec(T) and vec(T'), respectively obtained by piling up the entries of T and T', and to consider their ordering, based on the Lorenz curve (Joe, 1985).

In (Scarsini, 1990) a related approach is proposed, it is based on the partial ordering of the concentration curves obtained by the ratios $\rho_{ij}^* = n_{ij}/\hat{n}_{ij}$ $i=1,...,r$ and $j=1,...,s$, weighted by the independence frequencies $\hat{n}_{ij}$.

As it is well known, in case of independence all conditional distributions are similar to the marginal, so that a measure of concentration of the joint frequencies (or of the ratios ρ_{ij}^*) appears to be appropriate to measure the deviation from independence. Anyway, the n_{ij} arrangement (and the ρ_{ij}^* too) in non-decreasing order may lead to compare a cell in T_1 with a different cell in T_2. This is not meaningful in the study of dependence, where a pair (i,j) corresponds to a specific choice of the i-th category of A and the j-th category of B, so that n_{ij} and ρ_{ij}^* are conceptually related to this couple of indexes. Our approach to dependence orderings tries to overcome this problem.

The partial orderings introduced in (Forcina and Giovagnoli, 1987) are based on linear transformations on the tables, hence leading to the modification of one margin (or both): this proposal can not be useful in the chosen context of comparing tables in the class $\mathcal{T}(n_{i\bullet}; n_{\bullet j})$.

In this work a new approach is presented: first a frequency transfer among the cells of a two-way table which generates a decrease in dependence is introduced, successively a partial ordering that obey to these transfers is defined.

Let us consider tables T_1, T_2 and T_3, sharing the same marginal frequencies:

T_1: Frequencies n_{ij} before transfers

$A\backslash B$	b_1	b_2	b_3	
a_1	4	2	6	12
a_2	3	6	←15	24
a_3	5	16→	15	36
	12	24	36	72

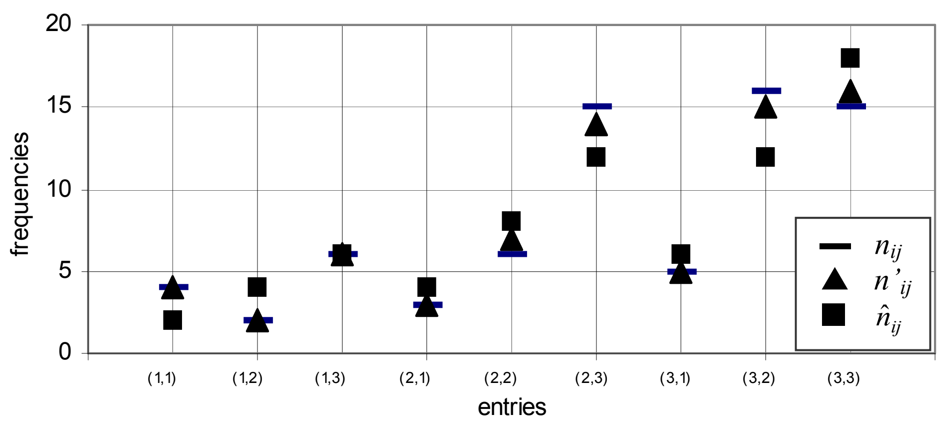
T_2: Frequencies $\hat{n}_{ij}$

$A\backslash B$	b_1	b_2	b_3	
a_1	2	4	6	12
a_2	4	8	12	24
a_3	6	12	18	36
	12	24	36	72

T_3: Frequencies n'_{ij} after transfers

$A\backslash B$	b_1	b_2	b_3	
a_1	4	2	6	12
a_2	3	7	14	24
a_3	5	15	16	36
	12	24	36	72

T_2 coincides with $T\{\hat{n}_{ij}\}$. T_3 is obtained from T_1 by transferring, as indicated by the arrows, one frequency from the cell in position (3,2) to the cell (3,3) and from (2,3) to (2,2). In the following graph the table cells are listed on the X-axis, while frequencies are on the Y axis:



The two transfers involve four cells and leave marginal data unchanged. Moreover, the two cells loosing a unit frequency, namely

(2,3) and (3,2), are such that $n_{ij} > \hat{n}_{ij}$. On the contrary, the two cells gaining a unit, i.e. (2,2) and (3,3), are such that $n_{ij} < \hat{n}_{ij}$. As a result of these transfers, as one can also realize from the graph, the modified frequencies n'_{ij} lay in an intermediate position between the initial n_{ij} and those of independency $\hat{n}_{ij}$. Furthermore, the signs of contingencies do not change, while their absolute values decrease. In other words, these transfers draw the table closer to the situation of independence, i.e. they decrease the 'directional dependence'. The qualifier *directional* emphasizes that contingency signs do not change.

The transfers from table T_1 to T_3 can be characterized also by evaluating the contingencies c_{ij} in T_1 and c'_{ij} in T_3:

Contingencies c_{ij} (before transfers)

$A \backslash B$	b_1	b_2	b_3
a_1	+2	−2	0
a_2	−1	−2	+3
a_3	−1	+4	−3

Contingencies c'_{ij} (after transfers)

$A \backslash B$	b_1	b_2	b_3
a_1	+2	−2	0
a_2	−1	−1	+2
a_3	−1	+3	−2

Obviously, one may consider T_1 as the transformation of T_3, obtained by two transfers (opposite to the previous ones), increasing directional dependence. Let us resume these considerations in a general definition:

Definition 2: *Frequency transfers that decrease the directional dependence*[2]

Let T be a table belonging to $\mathcal{T}(n_{i\bullet}\ ;\ n_{\bullet j})$; let $\{n_{ij}\}$ be the joint frequencies of T. Let (h,k), (h,g), (l,k) and (l,g) be four pairs of indexes such that $h \neq l,\ k \neq g$ and:

$$n_{hk} \geq \hat{n}_{hk} + 1; \qquad n_{lg} \geq \hat{n}_{lg} + 1;$$

[2] This definition has a strong analogy with the transformation suggested by Diaconis and Sturmfels (1998) and carried out over 4 cells, to generate a Markov chain in the class of all bivariate distributions with given margins.

$n_{hg} \leq \hat{n}_{hg} - 1;$ $n_{lk} \leq \hat{n}_{lk} - 1.$

Let T' be the table with joint frequencies n'_{ij} given by:

$n'_{hk} = n_{hk} - 1;$ $n'_{hg} = n_{hg} + 1;$

$n'_{lg} = n_{lg} - 1;$ $n'_{lk} = n_{lk} + 1;$

and $n'_{ij} = n_{ij}$ for all other pairs of indexes (i,j).

The transformation $T \rightarrow T'$ is called a frequency transfer that decreases the directional dependence.

In other words, directional dependence decreases when four joint frequencies became nearer to the corresponding independence frequencies by the effect of a frequency transfer acting over them.

Naturally, an analogue definition can be given for transfers that increase the directional dependence.

In dependence context, with the aim of defining an ordering relation, a suitable condition so that the distribution T shows more dependence than T' must be identified:

Definition 3: *Directional dependence ordering*

Let T and T' be two tables in $\mathcal{T}(n_{i\bullet}\,;\, n_{\bullet j})$ and let c_{ij} and c'_{ij} be their respective contingencies. In T there is higher directional dependence - between the variables A and B – with respect to T', if and only if, $\forall\, i,j \quad i{:}1,\ldots,r;\;\; j{:}1,\ldots,s:$

a) when both contingencies c_{ij} and c'_{ij} are not null, they have the same sign;
b) $|\, c'_{ij}\,| \leq |c_{ij}\,|$, with strict inequality for at least one pair (i,j).

The following notation will indicates it:

$$T' \prec_{\mathrm{DD}} T.$$

4. Numerical examples

In this section four classes $\mathcal{T}(n_{i\bullet}\,;\, n_{\bullet j})$ will be analyzed in detail, to emphasize the impact of the ordering relation $\prec_{DD}$. Although, for the sake of brevity, the chosen examples are very simple, the following methodology can be applied in the same way and with the same strength in depicting the degree of dependence of a given table T in its class $\mathcal{T}$.

Example 1: The class $\mathcal{T}\{n_{1.}=4, n_{2.}=6; n_{.1}=3, n_{.2}=7\}$

T_1

	b_1	b_2	
a_1	0	←4	4
a_2	3→	3	6
	3	7	10

T_2

1	←3	4
2→	4	6
3	7	10

T_3

2	←2	4
1→	5	6
3	7	10

T_4

3	1	4
0	6	6
3	7	10

$T\{\hat{n}_{lj}\}$

1.2	2.8	4
1.8	4.2	6
3	7	10

T_1, T_2, T_3, T_4 are the four tables of the chosen class, $T\{\hat{n}_{lj}\}$is the independence table. The arrows show the frequency transfers that allow the transformation from one table to the following one. The corresponding tables of contingencies are:

Cont. T_1

−1.2	+1.2
+1.2	−1.2

Cont. T_2

−0.2	+0.2
+0.2	−0.2

Cont. T_3

+0.8	−0.8
−0.8	+0.8

Cont. T_4

+1.8	−1.8
−1.8	+1.8

The prospect of comparisons between pairs of tables in the class turns out to be:

$T_2 \prec_{DD} T_1$

T_3 ? T_1; T_3 ? T_2;

T_4 ? T_1; T_4 ? T_2; $T_3 \prec_{DD} T_4$.

Among the six possible comparisons between pairs of tables, only two generate a ranking: T_2 shows lower directional dependence than T_1; the same is true for T_3 with respect to T_4. The symbol "?" between two tables indicates that they are not comparable on the bases of DD-ordering.

Example 2: The class $\mathcal{T}\{n_{1\cdot}=2, n_{2\cdot}=4, n_{3\cdot}=4; n_{\cdot 1}=3, n_{\cdot 2}=7\}$

T_1

	b_1	b_2	
a_1	0	2	2
a_2	0	←4	4
a_3	3→	1	4
	3	7	10

T_2

0	2	2
1	←3	4
2→	2	4
3	7	10

T_3

0	2	2
2	←2	4
1→	3	4
3	7	10

T_4

0	←2	2
3→	1	4
0	4	4
3	7	10

T_5

1	1	2
2→	2	4
0	←4	4
3	7	10

T_6

1	1	2
1→	3	4
1	←3	4
3	7	10

T_7

1	←1	2
0	4	4
2→	2	4
3	7	10

T_8

2	0	2
0	←4	4
1→	3	4
3	7	10

T_9

2	0	2
1	3	4
0	4	4
3	7	10

$T\{\hat{n}_{ij}\}$

0.6	1.4	2
1.2	2.8	4
1.2	2.8	4
3	7	10

The arrows show how successive transfers of two frequencies involving four cells can generate the enumeration of the tables in the class. With the aid of contingencies one can recognize the transfers that increase (decrease) dependence:

Cont. T_1

−0.6	+0.6
−1.2	+1.2
+1.8	−1.8

Cont. T_2

−0.6	+0.6
−0.2	+0.2
+0.8	−0.8

Cont. T_3

−0.6	+0.6
+0.8	−0.8
−0.2	+0.2

Cont. T_4

−0.6	+0.6
+1.8	−1.8
−1.2	+1.2

Cont. T_5

+0.4	−0.4
+0.8	−0.8
−1.2	+1.2

Cont. T_6

+0.4	−0.4
−0.2	+0.2
−0.2	+0.2

Cont. T_7

+0.4	−0.4
−1.2	+1.2
+0.8	−0.8

Cont. T_8

+1.4	−1.4
−1.2	+1.2
−0.2	+0.2

Cont. T_9

+1.4	−1.4
−0.2	+0.2
−1.2	+1.2

Out of the $\binom{9}{2} = 36$ available comparisons among pairs of tables, only four of them can be ranked according to the DD-ordering, namely:

$$T_2 \prec_{DD} T_1;\quad T_3 \prec_{DD} T_4;\quad T_6 \prec_{DD} T_8;\quad T_6 \prec_{DD} T_9.$$

Example 3: The class $\mathcal{T}\{n_{1.}=2,\ n_{2.}=3,\ n_{3.}=5;\ n_{.1}=2,\ n_{.2}=8\}$

T_1

	b_1	b_2	
a_1	2→	0	2
a_2	0	3	3
a_3	0	←5	5
	2	8	10

T_2

1	1	2
0	←3	3
1→	4	5
2	8	10

T_3

1→	1	2
1	←2	3
0	5	5
2	8	10

T_4

0	2	2
2→	1	3
0	←5	5
2	8	10

T_5

0	2	2
1→	2	3
1	←4	5
2	8	10

T_6

0	2	2
0	3	3
2	3	5
2	8	10

$T\{\hat{n}_{ij}\}$

0.4	1.6	2
0.6	2.4	3
1.0	4.0	5
2	8	10

Only two comparisons out of the $\binom{6}{2} = 15$ available produce a ranking between pairs of tables: $T_2 \prec_{DD} T_1$ and $T_5 \prec_{DD} T_4$.

In particular, note that, on the base of DD-ordering, one can not state that in T_1 there is higher directional dependence than in T_3, or in T_4, T_5, T_6, although T_1 is the table of absolute dependence (maximum bilateral dependence) between variables A and B.

Example 4: The class $\mathcal{T}\{n_{1.}=1, n_{2.}=2, n_{3.}=3; n_{.1}=1, n_{.2}=2, n_{.3}=3\}$

T_1

	b_1	b_2	b_3	
a_1	1	0	0	1
a_2	0	2→	0	2
a_3	0	0	←3	3
	1	2	3	6

T_2

1	0	0	1
0	1→	1	2
0	1	←2	3
1	2	3	6

T_3

1→	0	0	1
0	0	2	2
0	←2	1	3
1	2	3	6

T_4

0	1	0	1
0	0	←2	2
1	1→	1	3
1	2	3	6

T_5

0	1	0	1
0	←1	1	2
1→	0	2	3
1	2	3	6

T_6

0	1	0	1
1	0	←1	2
0	1→	2	3
1	2	3	6

T_7

0	1→	0	1
1	1	0	2
0	0	←3	3
1	2	3	6

T_8

0	0	1	1
1→	1	0	2
0	←1	2	3
1	2	3	6

T_9

0	0	1	1
0	2→	0	2
1	0	←2	3
1	2	3	6

T_{10}

0	0	1	1
0	1→	1	2
1	1	←1	3
1	2	3	6

T_{11}

0	0	1	1
0	0	2↓	2
1↑	2	0	3
1	2	3	6

T_{12}

0	0	1	1
1	0	1	2
0	2	1	3
1	2	3	6

$T\{\hat{n}_{ij}\}$

1/6	2/6	3/6	1
2/6	4/6	6/6	2
3/6	6/6	9/6	3
1	2	3	6

Among the $\binom{12}{2} = 66$ possible comparisons between the 12 tables, only in one case the ranking induced by the DD-ordering occurs: $T_2 \prec_{DD} T_1$.

5. Dependence Intensity

To increase the number of pairs of comparable tables according to their degree of dependence, the defining requirements must be weakened. The ordering $\prec_{DD}$ is very restrictive because it requires $r \times s$ comparisons (one for each pair of entries with equal row and column indexes in the two involved tables). Moreover, each comparison needs the evaluation of both the contingencies sign and their absolute value $|c_{ij}|$. A weaker criterion can be found by considering only the absolute value of contingencies:

Definition 4: *Intensity of dependence ordering (ID-ordering)*

Let T and T' be two tables in $\mathcal{T}(n_{i\bullet}; n_{\bullet j})$. In T there is higher intensity of dependence - between the variables A and B - with respect to T' if and only if: $|c'_{ij}| \leq |c_{ij}| \ \forall (i,j)$, with strict inequality for at least one pair (i,j). This situation is denoted by:

$$T' \prec_{ID} T.$$

If $|c'_{ij}| = |c_{ij}| \ \forall (i,j)$, the two tables are said to have the same intensity of dependence and this will be denoted by:

$$T' \approx_{ID} T.$$

Remark: The DD-ordering implies the ID-ordering.

Now it is useful to reconsider the previous four examples.

Example 1: The class $\mathcal{T}\{n_{1\bullet}=4, n_{2\bullet}=6; n_{\bullet 1}=3, n_{\bullet 2}=7\}$

One can check immediately that all tables are comparable w.r.t ID-ordering: $T_2 \prec_{ID} T_3 \prec_{ID} T_1 \prec_{ID} T_4$.

Indeed, for all 2×2 tables, as $|c_{11}| = |c_{12}| = |c_{21}| = |c_{22}|$, the ordering relation induced by Def. 4 is also a total ordering: hence all pairs of tables can be compared, so yielding a total ordering.

Example 2: The class $\mathcal{T}\{n_{1\bullet}=2, n_{2\bullet}=4, n_{3\bullet}=4; n_{\bullet 1}=3, n_{\bullet 2}=7\}$

Out of the 36 possible pairs of tables, only 18 of them produce a possible comparison, according to the ID-ordering. With reference to each chosen table, 8 comparisons arise, possibly classified in three ways: the chosen table is in a dominant position, is dominated, or it is not comparable. This information is summarized in the following prospect:

Table	dominates	no ranking with	is dominated by	total of comparisons
T_6	0	0	8	8
T_3	1	4	3	8
T_2	1	4	3	8
T_5	1	5	2	8
T_7	1	5	2	8
T_8	2	6	0	8
T_9	2	6	0	8
T_1	5	3	0	8
T_4	5	3	0	8

One can state that T_6 represents the minimum dependence in the given class because it is dominated, according to the ID-ordering, by all other tables in the class. In T_1 - and the same can be said for T_4 - there must be the maximum constrained dependence, because it dominates 5 tables, it is not dominated by any other table and it is not comparable with the remaining three.

Example 3: The class $\mathcal{T}\{n_{1\cdot}=2, n_{2\cdot}=3, n_{3\cdot}=5; n_{\cdot 1}=2, n_{\cdot 2}=8\}$

The ranking induced by the ID-ordering takes place in 9 out of the 15 pairs of tables. Each table is compared with the remaining 5 in the following way:

Table	dominates	no ranking with	is dominated by	total of comparisons
T_5	0	0	5	5
T_6	1	2	2	5
T_3	1	3	1	5
T_2	1	3	1	5
T_4	2	3	0	5
T_1	4	1	0	5

Table T_5 represents the lowest dependence, as it is dominated by all the other tables in the class. In T_1 there should be the highest dependence.

Example 4: ***The class*** $\mathcal{T}\{n_{1.}=1, n_{2.}=2, n_{3.}=3; n_{.1}=1, n_{.2}=2, n_{.3}=3\}$
With the ID-ordering the couples of comparable tables are 18, among the 66 possible pairs.

Table	dominates	no ranking with	is dominated by	total of comparisons
T_{10}	0	0	11	11
T_2	1	8	2	11
T_{11}	1	9	1	11
T_8	1	9	1	11
T_9	1	9	1	11
T_5	1	9	1	11
T_3	2	8	1	11
T_{12}	1	10	0	11
T_4	1	10	0	11
T_6	1	10	0	11
T_7	3	8	0	11
T_1	5	6	0	11

Table T_{10} is dominated by all other tables, hence it has the lowest dependence. T_1 should have the highest dependence as it dominates 5 tables, it is not dominated by any table and it is not comparable with the remaining 6 tables. Note that T_2 dominates T_{10}, hence T_2 has lower dependence in comparison with the other ten tables. Table T_7 should reflect a lower dependence than T_1 and a higher dependence in comparison with all other tables. The remaining eight tables have an intermediate degree of dependence between T_2 and T_7.

6. *Synthetic Dependence*

In the context of a partial ordering relation as $\prec_{DD}$ or $\prec_{ID}$, based on the comparisons of $r\times s$ contingencies, only a subset of pairs of tables in the same reference class belongs to the relation.

To enable the ranking of all pairs of tables, a total ordering - coherent with the partial orderings $\prec_{DD}$ and $\prec_{ID}$ - is needed. The new ordering will synthesize contingencies and hence it will be called synthetic dependence ordering.

Definition 5: *Synthetic dependence ordering (SD-ordering)*

Let T and T' be tables in $\mathcal{T}(n_{i\bullet}; n_{\bullet j})$. A synthetic dependence ordering $\prec_{SD}$ is defined on $\mathcal{T}(n_{i\bullet}; n_{\bullet j})$ if and only if the following three conditions are satisfied:

a) $T \prec_{SD} T'$ or $T \approx_{SD} T'$ or $T' \prec_{SD} T$;

b) $T \prec_{ID} T' \Rightarrow T \prec_{SD} T'$;

c) $T \approx_{ID} T' \Rightarrow T \approx_{SD} T'$.

Requirement a) states the nature of a total ordering, while conditions b) and c) assures the coherence with the ID-ordering.

From an operational point of view, synthetic dependence orderings can be defined by associating to each table of the class a scalar quantity. Our proposal is to evaluate this scalar quantity as a function f of contingencies $|c_{ij}|$, or of relative contingencies[3] $|\rho_{ij}|$:

$$f = f\left(|\rho_{11}|,\ldots,|\rho_{rs}|; n_{1\bullet},\ldots,n_{r\bullet}, n_{\bullet 1},\ldots,n_{\bullet s}\right),$$

considering the given marginal frequencies as parameters.

The following conditions:

a) f is not negative,

b) f is strictly increasing on each variable $|\rho_{ij}|$,

assure that f induces a (total) ordering coherent with Definition 5.

Two functions[4] satisfying these conditions are:

the weighted-arithmetic mean of $|\rho_{ij}|$ with weight $\hat{n}_{ij}$:

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[3] As that all the tables in a reference class share the same independence frequencies $\hat{n}_{ij}$, f can be based without distinction on $|c_{ij}|$, or on $|\rho_{ij}|$; the same is true for the orderings $\prec_{DD}$ and $\prec_{ID}$.

[4] More examples of well-known association indices that belong to this class of functions can be found in Polisicchio (2002).

$$\mathrm{M}_1\{|\rho|\} = \frac{1}{\mathrm{N}}\sum_j\sum_i|\rho_{ij}|\cdot\frac{n_{i\bullet}\times n_{\bullet j}}{n} = \frac{1}{\mathrm{N}}\sum_j\sum_i|c_{ij}|$$

the weighted-quadratic mean of $|\rho_{ij}|$ with weight $\hat{n}_{ij}$:

$$\mathrm{M}_2\{|\rho|\} = \left\{\frac{1}{\mathrm{N}}\sum_j\sum_i\left(|\rho_{ij}|\right)^2\cdot\hat{n}_{ij}\right\}^{1/2} = \left\{\frac{1}{\mathrm{N}}\sum_j\sum_i\frac{\left(c_{ij}\right)^2}{\hat{n}_{ij}}\right\}^{1/2} = \left\{\frac{1}{\mathrm{N}}\mathrm{X}^2\right\}^{1/2}$$

where X^2 denotes the classic Pizzetti-Pearson's statistic.

In conclusion, the two well-known indexes $M_1(|\rho|)$ and $M_2(|\rho|)$ induce, among the tables in a reference class, a synthetic dependence ordering $\prec_{SD}$. Moreover, $\prec_{SD}$ is coherent with the ordering $\prec_{ID}$ and therefore also with $\prec_{DD}$, yielding a hierarchy of orderings.

We will evaluate the indexes $M_1(|\rho|)$ and $M_2(|\rho|)$ on each table of the four examples to understand how the two indexes operate.

Example 1: The class $\mathcal{T}\{n_{1\bullet}=4,\ n_{2\bullet}=6;\ n_{\bullet 1}=3,\ n_{\bullet 2}=7\}$, whose independence table $T\{\hat{n}_{ij}\}$ is:

1.2	2.8
1.8	4.2

The following prospect reports, for each table in the given class, the absolute value of the relative contingencies, the mean values $M_1(|\rho|)$ and $M_2(|\rho|)$, and the index $\sigma(|\rho|)$.

As formerly remarked, in the case of 2×2 tables, the ID-ordering is already a total ordering, moreover the total orderings induced by $M_1(|\rho|)$ and $M_2(|\rho|)$ coincide with it.

We can also observe that $M_1(|\rho|)<M_2(|\rho|)$; this happens because the quadratic mean is always greater than the arithmetic one, with equality only in the case in which all $|\rho_{ij}|$ take the same value.

The prospect illustrates that in T_2, showing the minimum dependence, the observed frequencies differ from the independence frequencies, on the average, by a value corresponding to the 8% of their value. Variability among contingencies is low, as we can infer by the closeness between $M_2(|\rho|)$ and $M_1(|\rho|)$:

$$\sigma(|\rho|)=\{M_2^2(|\rho|)-M_1^2(|\rho|)\}^{1/2}=0.039$$

In table T_4, showing the maximum dependence compatible with the given margins, the observed frequencies differ from those of independence, on the average, by about the 72% of their value.

	$\lvert\rho_{ij}\rvert$		$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$\sigma(\lvert\rho\rvert)$
T_2	0.166	0.071	0.080	0.089	0.039
	0.111	0.048			
T_3	0.666	0.286	0.320	0.356	0.156
	0.444	0.190			
T_1	1.000	0.429	0.480	0.534	0.234
	0.666	0.286			
T_4	1.500	0.643	0.720	0.802	0.353
	1.000	0.429			

Example 2: ***The class*** $\mathcal{T}\{n_{1.}=2,\ n_{2.}=4,\ n_{3.}=4;\ n_{.1}=3,\ n_{.2}=7\}$ whose independence table $T\{\hat{n}_{ij}\}$ is:

0.6	1.4
1.2	2.8
1.2	2.8

In table T_6, showing the minimum dependence, the absolute difference from n_{ij} and $\hat{n}_{ij}$ equals, on the average, the 16% of the value of $\hat{n}_{ij}$; among the $|\rho_{ij}|$ a fair variability is observed.

Conversely, T_1 and T_4 are the tables with the maximum dependence. The deviations $|n_{ij}-\hat{n}_{ij}|$ are, on the average, the 72% of the value of $\hat{n}_{ij}$.

In T_8 and T_9, the index $M_1(|\rho|)$ is lower than in T_1 and T_4, while $M_2(|\rho|)$, in these four tables, takes the same value. This happens because variability among the $|\rho_{ij}|$ is greater in T_8 and T_9 than in T_1 and in T_4.

Note that the joint use of the indexes $M_1(|\rho|)$ and $M_2(|\rho|)$, or rather of $M_1(|\rho|)$ and $\sigma(|\rho|)$, allows to better discriminate between different situations:

	$\lvert\rho_{ij}\rvert$		$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$\sigma(\lvert\rho\rvert)$
T_6	0.667 0.167 0.167	0.286 0.071 0.071	0.160	0.218	0.148
T_3	1.000 0.667 0.167	0.429 0.286 0.078	0.320	0.408	0.253
T_2	1.000 0.167 0.667	0.429 0.078 0.286	0.320	0.408	0.253
T_5	0.667 0.667 1.000	0.286 0.286 0.426	0.480	0.535	0.236
T_7	0.667 1.000 0.667	0.286 0.429 0.286	0.480	0.535	0.236
T_8	2.333 1.000 0.167	1.000 0.429 0.079	0.560	0.802	0.574
T_9	2.333 0.167 1.000	1.000 0.078 0.429	0.560	0.802	0.574
T_1	1.000 1.000 1.500	0.429 0.429 0.643	0.720	0.802	0.353
T_4	1.000 1.500 1.000	0.429 0.643 0.429	0.720	0.802	0.353

In the following prospect, the number of tables having a higher, lower or equal value $M_i(|\rho|)$ (i = 1,2), is reported:

	Number of tables with synthetic dependence					
	lower than		equal to		higher than	
Table	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$
T_6	0	0	0	0	8	8
T_3	1	1	1	1	6	6
T_2	1	1	1	1	6	6
T_5	3	3	1	1	4	4
T_7	3	3	1	1	4	4
T_8	5	5	1	3	2	0
T_9	5	5	1	3	2	0
T_1	7	5	1	3	0	0
T_4	7	5	1	3	0	0

Example 3: ***The class*** $\mathcal{T}\{n_{1.}=2,\ n_{2.}=3,\ n_{3.}=5;\ n_{.1}=2,\ n_{.2}=8\}$ whose independence table $T\{\hat{n}_{ij}\}$ is:

0.4	1.6
0.6	2.4
1.0	4.0

The orderings induced by $M_1(|\rho|)$ and $M_2(|\rho|)$, among the tables of this class, besides confirming what was already clear by the ID-ordering, allows to rank all couples of tables. In particular, in T_5 there is the minimum dependence compatible with the given margins, and the lowest variability of contingencies. The comparison between T_1 and T_4 on the basis of both functions $M_1(|\rho|)$ and $M_2(|\rho|)$, allows to conclude that T_1 has higher dependence than T_4. All these considerations can be summarized by the following prospect:

	$\lvert\rho_{ij}\rvert$		$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$\sigma(\lvert\rho\rvert)$
T_5	1.000	0.250	0.160	0.289	0.241
	0.667	0.167			
	0.000	0.000			

	$\lvert\rho_{ij}\rvert$		$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$\sigma(\lvert\rho\rvert)$
T_2	1.500	0.375	0.240	0.433	0.360
	1.000	0.250			
	0.000	0.000			
T_6	1.000	0.250	0.400	0.500	0.300
	1.000	0.250			
	1.000	0.250			
T_3	1.500	0.375	0.400	0.520	0.332
	0.667	0.167			
	1.000	0.250			
T_4	1.000	0.250	0.560	0.764	0.520
	2.333	0.583			
	1.000	0.250			
T_1	4.000	1.000	0.640	1.000	0.768
	1.000	0.250			
	1.000	0.250			

The prospect also contains the following informations:

a) with reference to the index $M_1(\lvert\rho\rvert)$, T_3 and T_6 have the same degree of synthetic dependence;

b) the variability among the $\lvert\rho_{ij}\rvert$ is slightly higher in T_3 than in T_6.

The degree of dependence of a given table, with reference to that of all other tables in the same reference class, can be summarized by:

	Number of tables with synthetic dependence					
	lower than		equal to		higher than	
Table	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$
T_5	0	0	0	0	5	5
T_2	1	1	0	0	4	4
T_6	2	2	1	0	2	3
T_3	2	3	1	0	2	2
T_4	4	4	0	0	1	1
T_1	5	5	0	0	0	0

Example 4: The class $\mathcal{T}\{$ $n_{1.}=1,\ n_{2.}=2,\ n_{3.}=3;\ n_{.1}=1,\ n_{.2}=2,\ n_{.3}=3\}$ whose independence table $T\{\hat{n}_{ij}\}$ is:

1/6	2/6	3/6
2/6	4/6	6/6
3/6	6/6	9/6

As previously done, the following prospect summarize useful data about all tables in the given class:

	$\lvert\rho_{ij}\rvert$			$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$\sigma(\lvert\rho\rvert)$
T_{10}	1.000	1.000	1.000	0.444	0.601	0.405
	1.000	0.500	0.000			
	1.000	0.000	0.333			
T_2	5.000	1.000	1.000	0.550	1.014	0.852
	1.000	0.500	0.000			
	1.000	0.000	0.33			
T_6	1.000	2.000	1.000	0.611	0.882	0.636
	2.000	1.000	0.000			
	1.000	0.000	0.333			
T_5	1.000	2.000	1.000	0.667	0.833	0.499
	1.000	0.500	0.000			
	1.000	1.000	0.333			
T_8	1.000	1.000	1.000	0.667	0.833	0.499
	2.000	0.500	1.000			
	1.000	0.000	0.333			
T_4	1.000	2.000	1.000	0.722	0.882	0.507
	1.000	1.000	1.000			
	1.000	0.000	0.333			
T_{12}	1.000	1.000	1.000	0.722	0.882	0.507
	2.000	1.000	0.000			
	1.000	1.000	0.333			
T_9	1.000	1.000	1.000	0.944	1.054	0.469
	1.000	2.000	1.000			
	1.000	1.000	0.333			

	$\lvert\rho_{ij}\rvert$			$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$\sigma(\lvert\rho\rvert)$
T_3	5.000	1.000	1.000	0.944	1.202	0.744
	1.000	1.000	1.000			
	1.000	1.000	0.333			
T_{11}	1.000	1.000	1.000	1.000	1.000	0.000
	1.000	1.000	1.000			
	1.000	1.000	1.000			
T_7	1.000	2.000	1.000	1.056	1.118	0.367
	2.000	0.500	1.000			
	1.000	1.000	1.000			
T_1	5.000	1.000	1.000	1.222	1.414	0.711
	1.000	2.000	1.000			
	1.000	1.000	1.000			

The degree of dependence of a table, compared to those of all other tables – of the same reference class – can be summarized by:

	Number of tables with synthetic dependence					
	lower than		equal to		higher than	
Table	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$	$M_1(\lvert\rho\rvert)$	$M_2(\lvert\rho\rvert)$
T_{10}	0	0	0	0	11	11
T_2	1	1	0	0	10	10
T_6	2	3	0	2	9	6
T_5	3	1	1	1	7	9
T_8	3	1	1	0	7	9
T_4	5	3	1	2	5	6
T_{12}	5	3	1	2	5	6
T_9	7	8	1	0	3	3
T_3	7	9	1	0	3	2
T_{11}	9	6	0	0	2	5
T_7	10	9	0	0	1	2
T_1	11	11	0	0	0	0

The relative position induced by the two indexes is the same for three tables out of twelve: T_{10}, T_2 and T_1. It is almost the same for three tables - T_4, T_{12} and T_9 - and it is rather different for the remaining tables.

7. Conclusions

This work proposes the introduction of new partial and total dependence orderings to allow a deeper understanding of some aspects of dependence. The definition of the directional dependence ordering $\prec_{DD}$ takes into account the signs and the absolute values of $r \times s$ contingencies. This notion is natural enough and in a sense more intuitive than earlier proposals for dependence ordering considered in literature. In addition, a special kind of frequency transfer, defined over four cells in a bivariate table and increasing dependence, is studied. Starting from the orthant in which the given table *T* lays, by a series of transfers increasing the directional dependence, one can locate a situation (a table) with maximum dependence: the uniquely identified distributive independence situation is then counterbalanced by several situations of maximum dependence in orthants.

Starting from the partial ordering $\prec_{DD}$, and successively weakening the ranking criterion, a hierarchy of orderings is defined, till yielding a (total) synthetic dependence ordering. Section 7 shows that various functions of $|\rho_{ij}|$ (and of marginal frequencies) inducing total dependence orderings, can be used. The choice of the particular function *f* to be adopted in order to obtain a synthetic dependence index must therefore be based on further properties of *f*, besides its capability to generate total orderings useful in dependence context. The same Section explains the reasons for which it is advisable to use jointly Mortara's index $M_1(|\rho|)$ and Pearson's $M_2(|\rho|)$, since $M_1(|\rho|)$ provides the order of magnitude of the $|\rho_{ij}|$, and the difference between $M_2(|\rho|)$ and $M_1(|\rho|)$ gives their measure of variability. Perhaps, for descriptive purposes, it would be better to employ $\sigma(|\rho|)$ together with $M_1(|\rho|)$. The idea to evaluate the degree of dependence in a given table *T*, by considering how many other tables of its reference class have a lower, equal or greater value of the chosen synthetic index in comparison to that of *T*, appears to be worthy of future developments. Further work should also deal with the characterization of minimal and maximal tables with respect to the DD and the ID partial orderings.

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