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Non-Linear Dynamics and Evaluation of forecasts using High-Frequency Time Series

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Summary: In the present paper we evaluate the performance of a non linear parametric model in forecasting high-frequency data. In particular we consider the TAR-ARCH model (Li and Lam ,1995) to fit and forecast the daily and 5-minute returns of the Mibtel Stock Index.

Keywords: TAR Models, ARCH Models, High Frequency Data, Forecasts

1. Introduction

In the last few years the analysis of high frequency data has been widely developed in the literature. Among the others, Andersen, Bollerslev *et al.* (1998) have shown that the heteroscedasticity is better cached as the sample frequency grows and the use of intra-daily data allow to obtain more accurate forecasts.

The aim of the present paper is to evaluate the performance of a particular non-linear parametric model in forecasting time series at frequencies higher than daily.

More precisely, we consider a threshold type non-linear model (Tong, 1990), that allows for asymmetry in the conditional mean, combined with a non linear model for the changing conditional variance (TAR-ARCH, Li and Lam, 1995).

The prediction ability of the TAR-ARCH model is investigated for different sampling frequencies. Therefore we analyse the empirical behaviour of the daily and intra-daily Mibtel Stock Market Index.

The structure of the paper is the following: section 2 illustrates some issues connected with high frequency data and briefly reports the modelling procedure; section 3 shows the estimated models and their forecasting performances and gives some concluding remarks.

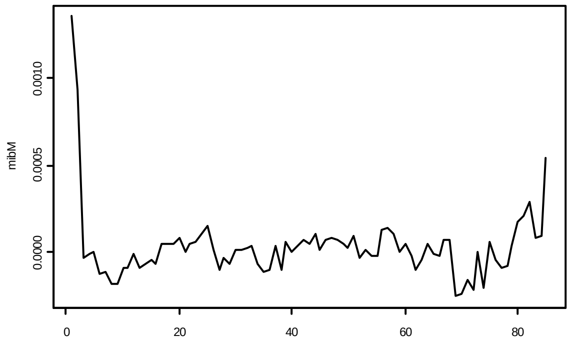
2. Modelling High Frequency Data

The availability of financial data at frequencies higher than daily allows for a wide range of issues in financial market. In the recent literature many authors have investigated the empirical evidence of high frequency financial time series (among the others, Andersen and Bollerslev, 1998b; Goodhart and O'Hara, 1997). The low frequency data often mask the real features of markets and their high volatility which is often not adequately recognised.

The high frequency data give a new opportunity in the analysis of volatile time series and the advantage of their use in estimating and forecasting volatility is pointed out in different empirical works (Andersen, 2000). Employing high frequency data also give the opportunity to uncover the microstructure pattern of the market and the effects of intra-day seasonality.

These features are well recognised in the high frequency Mibtel returns. The pronounced periodicity of the Mibtel returns during the trading day establishes that the volatility is higher at the beginning and at the end of the trading day and falls quite rapidly to lower levels during the mid-day assuming the typical U shaped pattern (Andersen and Bollerslev, 1997). In Fig.1 the average 5-minute returns of the Mibtel are shown. Each value is obtained as $X_{t,h} = N^{-1}\sum_{i=1}^{N} Y_{i,h}$, where N is the total number of trading days from January 2 to July 30, 1998 and h is the intra-daily observation, h=1,...,85 (a more detailed description of the data is available in section 3).

Figure 1. Intra-day average of the Mibtel returns



Theoretical and empirical evidence show how the response of stock market index exhibits a pronounced asymmetric cyclical behaviour due to the different reaction to positive and negative shocks. The piece-wise linear threshold model by Tong (1983) allows to modelling asymmetry in the conditional mean. In order to take into account, also, the changing conditional variance of many financial time series Li and Lam (1995), following a suggestion in Tong (1990), introduced an ARCH specification (Engle, 1982), defined on the TAR model residuals, obtaining the so called TAR-ARCH model.
Let $\{Y_t\}$ be a time series generated by a stationary process, a TAR-ARCH $(l, p_1, p_2, ..., p_l; q)$ model is given by:

$$Y_t = a_0^{(j)} + \sum_{i=1}^{p_j} a_i^{(j)} Y_{t-i} + \varepsilon_t \qquad r_{j-1} < Y_{t-d} \le r_j \tag{1}$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}$$

for $j=1,2,...,l$, the threshold values, $\{r_0, r_1, r_2,..,r_l\}$, are such that, $r_0<r_1<...<r_l$, $r_0 = -\infty$ and $r_l = +\infty$ with $R_j = (r_{j-1}, r_j]$ and $\{e_t\}$ is i.i.d. with zero mean and conditional variance h_t.
For the testing and identification procedure we follow the proposal of Tsay (1989), further developed and extended in Tsay (1998), based on

the value of an arranged regression on the increasing order of the threshold variable.

3. Forecasting daily and intra-daily Mibtel Stock Index

In order to assess the predictive ability of the SETAR-ARCH model, we have analysed the Mibtel Index of the Italian stock market from January 2, 1998 through July 30, 1998 observed at different sampling frequencies.

Following the standard practice, we have analysed the returns time series data obtained as $DY_{t,h} \equiv log(Y_{t,h}/Y_{t-1,h})$ where t refers to the trading day and h refers to the intra-daily observation of the high-frequency series.

In the following two subsections we show the results of the analysis of the daily and intra-daily Mibtel Stock Index and evaluate the forecasting performances of the SETAR-ARCH model. For the intra-daily data we consider the AR-ARCH model as a benchmark for the forecast comparison.

3.1 Daily Mibtel Stock Index

For the daily Mibtel series the sample period include 148 observations. The time plots of the daily levels and of the transformed Mibtel Index series are given in Fig.2a) and 2b) respectively.

The model has been fitted to a sample of 132 daily returns (from January 2 to July 10, 1998) living the next 15 (from July 13 to July 30) to assess the model forecasting performance.

The histogram of the returns in Fig.3a) shows the high kurtosis of the data distribution, confirmed by the value of the kurtosis index, K=4.6652, and the fatter tails than the normal distribution.

The subsequent analysis of data highlights that the seeming whiteness shown by the correlogram of the autocorrelation function (ACF) hides a non linear structure of the data generating process which is further investigated by means of two tests: the Tsay's linearity test (Tsay, 1989) and the ARCH-LM test (Engle, 1982).

The results of the former test, which evaluates the threshold type non linearity, are shown in Tab.1 where the hypothesis of linearity is strongly rejected for different time delays.

Figure 2. Time plots: a) daily Mibtel; b)DMib

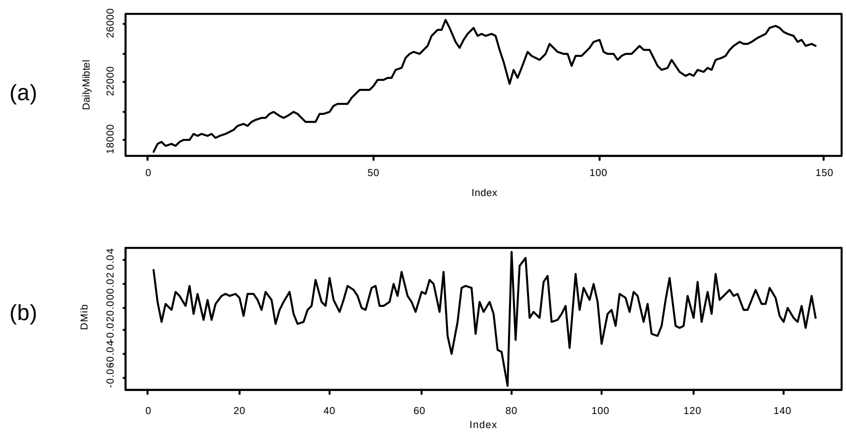
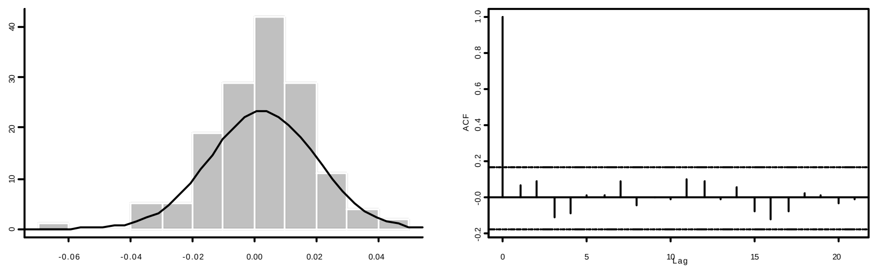


Figure 3. Correlogram of the ACF and histogram with normal density



(a) (b)

It is commonly know that the behaviour of the stock market returns is influenced by what has happened in the previous days. This suggests to consider as threshold variable a lagged value of the returns which leads to the identification of a SETAR (Self Exciting Threshold Autoregressive Model; Tong, Lim, 1980).

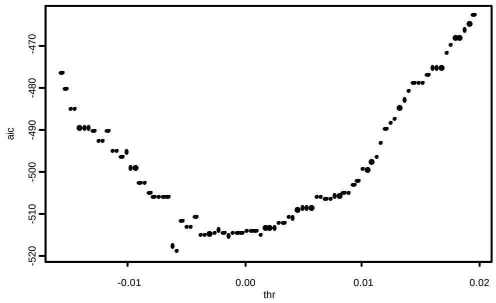
The heteroscedasticity of the series is also confirmed by the ARCH-LM test which equals to 32.772 (p-value 0.0000) at lag 1.

Table 1. Tsay's linearity test

d	1	2	3	4	5
Test	2.535	1.109	12.435	7.100	5.944
d.f.	1	1	1	1	1

The Tsay's linearity test indicates that the test statistics is maximised at the threshold lag d=3, suggesting this as the optimal choice for the delay of the threshold variable. Setting the AR order to lie in the range [1,2], given d=3 and the number of regimes $j \in [2,3]$, we select the threshold value $\hat{r}_{t-3} = -0.0057$ that gives the minimum Akaike Information Criteria (AIC), as shown in Fig.4.

Figure 4. Threshold values versus AIC



The final estimated model is the following SETAR(2;1,1)-ARCH(1) (standard deviations of the least squares estimates are given in parenthesis):

$$Y_t = \begin{cases} \underset{(0.0027)}{0.0064} - \underset{(0.129)}{0.4802}\, DMib_{t-1} + e_t & DMib_{t-3} \leq -0.0057 \\ \underset{(0.0994)}{0.4678}\, DMib_{t-1} + e_t & DMib_{t-3} > -0.0057 \end{cases}$$

$$h_t = \underset{(2.84E-05)}{0.000169} + \underset{(0.12677)}{0.24396}\, e_{t-1}^2$$

The residual analysis has been performed on the basis of the Ljung – Box (1978) and the Jarque-Bera (1980) tests. As shown in Tab.2 the SETAR-ARCH residuals have not a significant structure, even if the presence of residual kurtosis in the series affects the results of the Jarque-Bera test.

Table 2. Residual Tests

	L-B(20)	Jarque-Bera
Test	9.0155	6.0074
p-value	0.983	0.05

The forecasting performance of the SETAR-ARCH model has been assessed on the basis of different loss functions that achieve to the evaluation of the forecasts of the conditional mean and conditional variance differently.

The Mean Square Error (MSE), the Root Mean Square Error (RMSE) and the Mean Absolute Error (MAE) indexes, in this context, evaluate the magnitude of the conditional mean forecast errors; the Proportion of Correct Sign (PCS) index (Brooks, 1997) evaluates the ability of the model to predict direction changes irrespective of their magnitude; the LL and the LAE index (Pagan and Schwert, 1990) are two loss functions which penalise volatility forecasts asymmetrically and are defined as:

$$LL = \frac{1}{T}\sum_{t=1}^{T}\left[\ln(e_{t+k}^2) - \ln(\hat{h}_t(k))\right]^2 \qquad LL = \frac{1}{T}\sum_{t=1}^{T}\left|\ln(e_{t+k}^2) - \ln(\hat{h}_t(k))\right|$$

Table 3. Forecast Evaluation of the SETAR-ARCH model

Conditional Mean				
	$MSE\times10^6$	$RMSE\times10^2$	$MAE\times10^2$	PCS
SETAR-ARCH	0.61299	0.02476	0.01841	0.5333

Conditional Variance		
	LL	LAE
SETAR-ARCH	7.31691	1.83834

where $\hat{h}_t(k)$ is the predicted variance and e_{t+k} is generated taking into account the distribution of the errors of the conditional mean model. The forecast evaluation indexes are presented in Tab.3.

3.2 Intra-daily Mibtel Stock Index

The growing sampling frequency has a very strong impact on the distributional properties of the data. In this section we show the results of the analysis of the intra-daily Mibtel returns series. They are the 5 minute Mibtel Index from 10 a.m. to 5 p.m. from January 2, 1998 to July 30, 1998 which consists of 85 intra daily-data for a total of 12.535 observations.

The models are fitted to a sample of 11.267 observations and the data relative to the last 15 trading days have been used to assess the forecasting performances of the fitted models.

The intra-daily Mibtel time plot is given in Fig.5a) and its returns are presented in Fig.5b). The latter plot shows the high volatility of the intra-daily returns which have the typical pattern of high-frequency financial time series. Some summary statistics of the returns are given in Tab.4 where the excess kurtosis indicates the necessity of fat tailed distributions to describe the data. The correlogram of the ACF and of the Partial ACF are presented in Fig.6.

Following the same steps of the previous analysis, the non-linearity has been investigated by means of the Tsay's test and the ARCH-LM test.

Table 4. Summary Statistics

Mean	Max	Min
1.46E-05	0.009374	-0.011102
S.D.	Skewness	Kurtosis
0.000655	0.137058	47.18535

Figure 5. Time plots: a) daily Mibtel; b)DMib

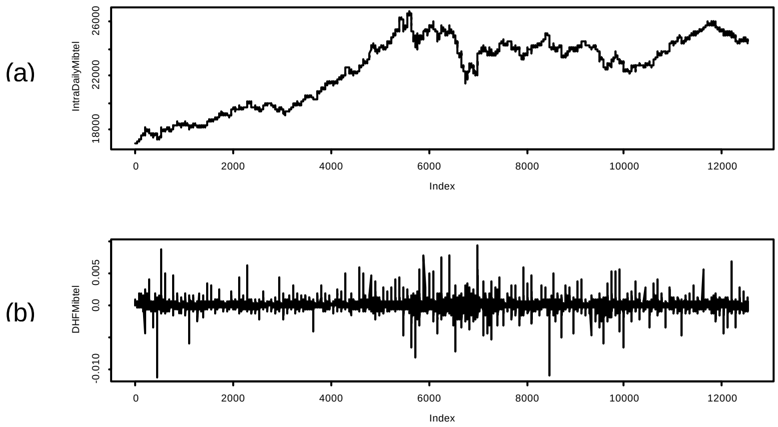
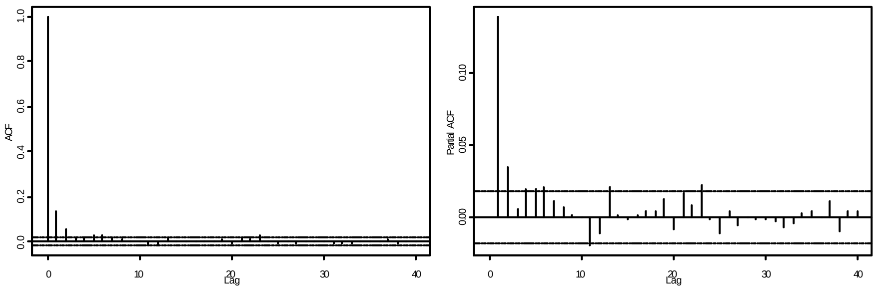


Figure 6.   Correlogram of DMib ACF and PACF



In both cases the linearity is not accepted and the Tsay's test (Tab.5) strongly suggests threshold non-linearity for the conditional mean model, with the highest value for d=1.

Following the identification procedure proposed by Tsay (1989), the threshold values are selected in order to minimise the AIC with $j \in 2,3$, given the delay d=1 and the range of the autoregressive order $p \in [1,6]$.

The further investigation of the series highlights a conditional variance structure which is modelled by means of a GARCH(2,1) model. The least squares estimates of the coefficients of the refined

SETAR(3;1,2,4)-GARCH(2,1) model and the estimated threshold values are given in Tab.6.

Table 5. Non-linearity tests

Tsay's test				
d	1	2	3	4
Test	21.485	19.154	3.980	10.764
d.f.	5	5	5	5
Engle ARCH-LM test				
Test	117.2328	*p-value*		0.0000

The analysis of the residuals of the SETAR-GARCH model does not indicates significant correlations in the residuals and in the squared residual series which have been investigated by means of the Ljung-Box test which equals 27.813 (p-value 0.114) and 1.7979 (p-value 0.999) respectively, for up to 20-th order serial correlation.

Table 6. Estimated Coefficients SETAR-GARCH model

SETAR					
	a_0	a_1	a_2	a_3	a_4
I	-0.00004 (0.00001)	0.10963 (0.01898)			
II	0.00002 (0.00001)		0.07971 (0.02235)		
III	0.00004 (0.00002)	0.12025 (0.02197)	0.04575 (0.01911)		0.06166 (0.02009)
Threshold 1		-0.0001	***Threshold 2***		0.00017
GARCH					
	c	a_1	a_2	b_1	
	6.87E-08 (2.98E-08)	0.13333 (0.00966)	0.04444 (0.00933)	0.53333 (0.01213)	

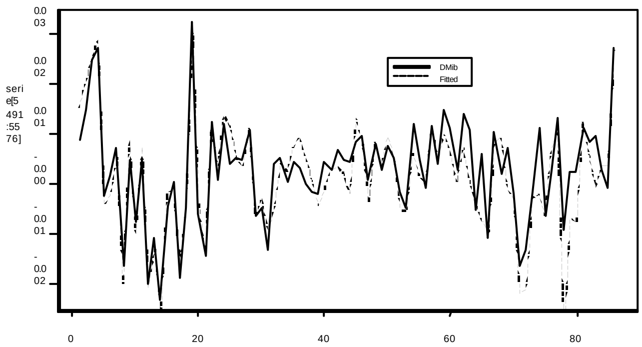
In order to evaluate the forecasting performance of the fitted SETAR-GARCH model we have fitted an AR(1)-GARCH(1,1) model to the intra-daily returns. The coefficient estimates of the AR-GARCH model are given in Tab.7.

Table 7. Estimated Coefficients AR-GARCH model

	f_0	f_1	
AR	1.45E-05	0.14409	
	(7.14E-06)	(0.00932)	
	a_0	a_1	b_1
GARCH	1.31E-07	0.15	0.6
	(2.49E-08)	(0.00828)	(0.00925)

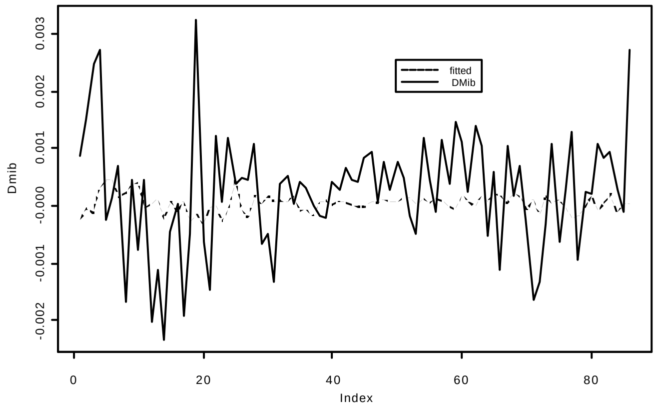
In Fig.7 and Fig.8 we have compared the fit obtained from the SETAR-GARCH and the AR-GARCH models respectively with the original data of one trading day.
Fig.7 show that the high volatility of the opening of the stock market is adequately captured by the SETAR-GARCH model which appears less sensible to the random pattern of the last 20 observations which correspond to the closing time of the market.

Figure 7. SETAR-GARCH model fitted to data



In order to evaluate the forecasting performance of the SETAR-GARCH model with respect to the AR-GARCH, we have used different loss functions (presented in section 3.1) for the conditional mean and the conditional variance.

Figure 8. AR-GARCH model fitted to data



A further aim of the forecast assessment study is the evaluation of the multi-steps ahead forecasts generated by means of the conditional mean models. In this contest we have generated the 6-steps ahead forecasts of the intra-daily returns (which correspond to 30 minute ahead forecasts) for both the models under study.

Table 8. Conditional Mean Forecast Evaluations

		$MSE\times10^6$	$RMSE\times10^2$	$MAE\times10^2$	PCS
AR-ARCH	1-step	0.2674	0.05171	0.02961	0.53549
	6-steps	0.2665	0.05163	0.02983	0.48580
SETAR-ARCH	1-step	0.2709	0.05205	0.03018	0.49211
	6-steps	0.2664	0.05161	0.02982	0.4874

Table 9. Conditional Variance Forecast Evaluations

	LL	LAE
AR-ARCH	14.87057	3.12506
SETAR-ARCH	4.49392	1.96743

The results of the forecast assessment shown in Tab.8 and Tab.9 highlight that the good fitting performances of the SETAR-GARCH

model do not guarantee a good performance of the forecasting procedure.
The forecasting performances of the conditional mean of the SETAR-GARCH and AR-GARCH models imply that:

- the one-step ahead forecast performances measured in terms of MSE, RMSE, and MAE are similar for both models;
- the six steps ahead forecast performance of the SETAR-GARCH model, measured in terms of the same loss functions, is slightly better than that of the AR-GARCH model;
- the Proportion of Correct Sign of the AR-GARCH model is relatively better for the one-step ahead forecast but in the 6-steps ahead case the SETAR-GARCH model slightly outperforms the previous model.

These results show that the high volatility of the intra-daily series influences the forecast performance of the conditional mean of both models.
The results of the conditional variance forecast evaluation are completely different (Tab.9). In this case, we observe a substantial improvement of the predictive performance of the SETAR-GARCH model with respect to the AR-GARCH model in terms of LL and LAE indexes.
Fig.9 and Fig.10 show the one-step ahead forecasts of the SETAR-GARCH and AR-GARCH models within one trading day. The narrower prediction bounds in Fig.9, with respect to Fig.10, are

Figure 9. One step ahead SETAR-GARCH forecasts

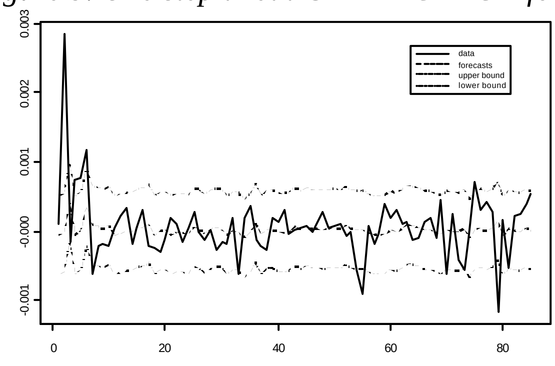
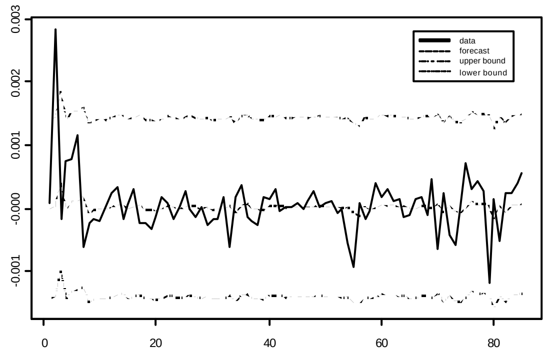


Figure 10. One step ahead AR-GARCH forecasts



evident. This confirm, in this case, the non trivial reduction of the forecast uncertainty obtained using the SETAR-GARCH model instead of the AR-GARCH model.

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