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Inference in threshold autoregressive conditional heteroscedastic models: a resampling approach

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Summary: In this paper some inferential problems arising in the estimation of TAR-ARCH and DTARCH models are investigated. In particular the subseries method is proposed as a diagnostic tool for describing the sampling distribution of the parameters estimators. The results of a Monte Carlo simulation experiment are illustrated and discussed.

Key words: threshold models, sampling distribution, subseries approach, time series

1. Introduction

In the last two decades modelling non-linear time series has received considerable attention and a number of non-linear models have been proposed in literature. A comprehensive review of many of these can be found in Tong (1990) and more recently in Tjøstheim (1994).One of the most widely used classes of non linear time series models is the family of Threshold Autoregressive (TAR) model first proposed by Tong (1978) and in a more complete version by Tong and Lim (1980). The piecewise linear autoregressive conditional mean of this structure allows to model asymmetry, jump phenomena and limit cycles.

It is recognised that the behaviour of many real-life time series, such as financial time series, shows a changing conditional variance.

One popular approach in modelling heteroscedasticity is the ARCH (Autoregressive Conditional Heteroscedastic) model proposed by Engle (1982) and in a generalised version by Bollerslev (1986), where the conditional variance is modelled as a linear function of past squared errors.

Tong (1990) proposed to combine the use of TAR model with an ARCH specification in order to simultaneously capture the non-linear dependence in the mean and the changing conditional variance. This proposal was recently elaborated by Li and Lam (1995) and further extended by Liu, Li and Li (1997), who analyse the identification and estimation strategy of TAR-ARCH and DTARCH models.

The flexibility of this non-linear structures allows to model very complex phenomena but makes particularly hard to derive probabilistic and statistical properties. Liu et al. (1997) determined, under appropriate assumptions, the stationarity and ergodicity conditions for a DTARCH model. The problem remains open for the properties of the sampling distribution of the estimators, where the classical asymptotic results can't be easily extended.

In this context resampling techniques for dependent data can be usefully applied. In particular, we propose a procedure based on the subseries approach (Sherman, 1997) in order to describe the sampling distributions of the parameters estimators of TAR-ARCH and DTARCH non-linear models.

The paper is organised as follows. The next section briefly describes the models considered. Section three illustrates the subseries approach procedure for threshold autoregressive models. The results of a Monte Carlo simulation experiment are shown in section four. The last section concludes and gives some final comments.

2. The Threshold Autoregressive Conditional Heteroscedastic Models

Tong (1990) suggested for the first time to use a threshold model with changing conditional variance in order to combine the advantages of the threshold principle and of the ARCH innovations.

The so called TAR-ARCH model was further developed and applied by Li and Lam (1995), it has a piecewise linear autoregressive structure for the conditional mean and a conditional variance that is a linear function of squared past disturbances.

Let $\{Y_t\}$ be a time series generated by a stationary process, a TAR-ARCH(k, $p_1 \ldots p_k$; q) model is given by:

$$Y_t = a_0^{(j)} + \sum_{i=1}^{p_j} a_i^{(j)} Y_{t-i} + \varepsilon_t \qquad r_{j-1} < Y_{t-d} \le r_j \tag{1}$$

$$h_t = \alpha_o + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

for $j=1,2,\ldots,k$, the *threshold values*, $\{r_0, r_1, r_2, \ldots, r_k\}$, are such that $r_0 < r_1 < \ldots < r_k$, $r_0 = -\infty$ and $r_k = +\infty$ with $R_j = (r_{j-1}, r_j]$ and $\{\varepsilon_t\}$ is i.i.d. with zero mean and conditional variance h_t.

In this structure the conditional variance depends only on the amplitudes of the shocks, the different effects of positive and negative shocks is not considered. Some authors pointed out, particularly in the financial framework, that the sign of the errors can have different influence on the volatility.

To introduce the asymmetry both in the level and in the conditional variance, Li and Li (1996) proposed to use a threshold principle also in the ARCH residuals.

A time series $\{Y_t\}$ follows a DTARCH ($k_1, p_1, \ldots, p_{k1}; k_2, q_1, \ldots, q_{k2}$) (Double Threshold Autoregressive Conditional Heteroscedastic) model if:

$$\begin{aligned} Y_t &= a_0^{(j)} + \sum_{i=1}^{p_j} a_i^{(j)} Y_{t-i} + \varepsilon_t \qquad r_{j-1} < Y_{t-d} \le r_j \\ h_t &= \alpha_0^{(v)} + \sum_{i=1}^{q_v} \alpha_i^{(v)} \varepsilon_{t-i}^2 \qquad c_{v-1} < Y_{t-b} \le c_v \end{aligned} \tag{2}$$

where $\{\varepsilon_t\}$ is i.i.d. with zero mean and var($\varepsilon_t|I_{t-1}$)=h_t, the *threshold values*, r_j and c_v, are such that $r_0<r_1<r_2<..<r_{k_1}$, $r_0= -\infty$ and $r_{k_1}=+\infty$, $c_0<c_1<...<c_{k2}$, $c_0= -\infty$ and $c_{k2}= +\infty$, with $j=1,2,...,k_1$, $v=1,2,....,k_2$, d and b are the *delay parameters* respectively for the conditional mean and for the conditional variance.
To ensure a well-defined models the following regularity conditions must be considered:

- the time series $\{Y_t\}$ is stationary and ergodic;
- $E(Y_t^2)<\infty$ and $E(\varepsilon_t^4)<\infty$;
- all parameters in the conditional variance model are positive or non-negative, $\alpha_0^{(v)}>0 \quad \alpha_i^{(v)}\geq 0$ for $i=1,2,...,q_v$ e $v=1,2,...,k_2$
- $(\varepsilon_{t-1}^2,\varepsilon_{t-2}^2,\ldots,\varepsilon_{t-q_v}^2)$ is linearly independent, for $v=1,2,..$, k_2
- characteristic roots of the autoregressive polynomial of the conditional mean in each regime are such to satisfy the stationarity conditions for an AR process;
- let $a^{(j)}=(a_0^{(j)},a_1^{(j)},\ldots,a_q^{(j)})^T$ and $\alpha^{(v)}=(\alpha_0^{(v)},\alpha_1^{(v)},\ldots,\alpha_q^{(v)})^T$, then $a^{(j)}\neq a^{(j')}$ and $\alpha^{(v)}\neq\alpha^{(v')}$, for $j\neq j'$ and $v\neq v'$

Assuming that the regularity conditions hold, Liu et al. (1997) derived the likelihood function for a DTARCH with only two regimes for the conditional mean and the conditional variance. Wherever the same results can be extended to a more complex structure, some problems arise in the definition of the sampling properties of the parameters estimators. They also give some asymptotic results for the consistence of the MLE estimators, but only under the strong assumption of known threshold values and delay parameters. If the parameters are unknown, as it is often the case in real situations, the problem becomes much more complicated. The non-differentiability of the likelihood function prevents the usage of standard asymptotic results.

In order to obtain an approximation of the sampling distribution of the parameters estimators, resampling procedures can be used in this context.

3. Subsampling in threshold models

In the recent literature much attention is devoted to the extension of resampling methods to dependent data problems, such as time series analysis.

In this framework some results were shown by Kunsch (1989) who proposed to divide the observed time series into blocks in order to capture the dependence in the original series and derived the properties of the so called Moving Blocks Jackknife and Moving Blocks Bootstrap (MBJ and MBB). The moving blocks procedure has been studied in the mean time by Liu and Singh (1992) and Politis and Romano (1994).

An alternative approach for dependent data was proposed by Sherman and Carlstein (1996) and extended by Sherman (1997). They use subseries of shorter length taken as replicates of the original data structure. The Subseries Approach (SA) can be used to estimate the distribution function of an arbitrarily complicated statistics when its theoretical derivations is analytically intractable. The SA can be seen as a simple diagnostic tool, based on a finite data set, without requiring any theoretical analysis by the user.

We propose to use the SA to approximate the sampling distribution of the MLE parameters estimators of TAR-ARCH and DTARCH models where, as we have seen, the complexity of the structure and problems related to the identification of some elements, such as the threshold and the delay, make very difficult deal with standard inference results.

Let $Y_n=(Y_1,\ldots,Y_n)$ be a time series of n observations generated by a strictly stationary random process. Starting from Y_n, consider the blocks of consecutive observations $Y_l^i=(Y_i,Y_{i+1},\ldots,Y_{i+l-1})$, for $i=0,\ldots,(n-l)$ with $l<n$. Let $s_n = s_n(Y_n)$ be the statistic of interest to be calculated on the entire series, we can use the behaviour of the subseries replicate $s_l^i = s_l(Y_l^i)$ to estimate the future of the unknown distribution of s_n.

In particular we can obtain a graphical representation of the sampling distribution by the empirical distribution function of the subseries replicates given by:

$$\tilde{F}(y) = \frac{\sum_{i=0}^{n-l} I[s_l^i \le y]}{n-l+1} \tag{3}$$

The principal problem to be taken into account in applying the subseries approach is, like in the moving blocks procedure, to chose the length l so that the subseries retain almost the same dependence structure of the original time series Y_n. Some general results in the determination of the length was presented in Kunsch (1989). In the subseries context Sherman et al. (1996) and Sherman (1997) showed that $l(n)=[cn^{\gamma}]$ for any fixed $c>0$ and $\gamma\in(0,1/2)$. They suggest, based on theoretical examples, that an optimal choice is $l=(2n^{1/3})$.

An alternative procedure to determine $l(n)$, both in the MBB and in SA context, was recently proposed by La Rocca and Vitale (1999). In the determination of the length of blocks they take into account the maximum degree of dependence in Y_n.

For an autoregressive linear structure they suggested that an optimal choice for the length of the subseries is given by:

$$l(n)\ge n\phi^2 \tag{4}$$

where ϕ is the maximum characteristic root that dominates the strength of dependence.

In the threshold model the non-linearity is captured by a piecewise linear structure, since the process follows an autoregressive linear model in each regime. The strength of dependence in each regime is still dominate by the AR maximum characteristic root.

Therefore we can easily extend to the TAR family the results of the linear context and calculate the subseries length as:

$$l(n) = \sum_{j=1}^{k} l_j(n_j) \tag{5}$$

where $l_j(n_j)$ is the length calculated in each regime given by:

$$l_j(n_j) \geq n_j \phi_j^2 \tag{6}$$

with n_j being number of observations in the j-th regime, $\sum_{j=1}^{k} n_j = n$, and ϕ_j the maximum characteristic root of the AR process in the j-th regime. If we consider an ARCH structure in the error term the strength of dependence being dominate by the maximum characteristic root of the AR structure for both the conditional mean and the conditional variance. Therefore in the (6) we have $\phi = \max(\phi_T, \phi_A)$, with ϕ_T and ϕ_A the maximum characteristic roots for the AR structure respectively for the conditional mean and variance.

4. Simulation Experiment

In order to obtain a graphical approximation of the sampling distribution function of the parameters estimators of threshold autoregressive models with conditional heteroscedasticity and evaluate their bias, we perform the subseries approach on simulated time series. The Monte Carlo simulation experiment is organised as follows.

We generate non-linear time series starting from a DTARCH model, whose flexible structure allows to contain all the simpler forms. The simulated model is a DTARCH(1,1;1,1):

$$\begin{cases} Y_t^{(1)} = -0.6Y_{t-1}^{(1)} + \varepsilon_t & Y_{t-1} \leq 0 \\ Y_t^{(2)} = 0.4Y_{t-1}^{(2)} + \varepsilon_t & Y_{t-1} > 0 \end{cases}$$

$$\begin{cases} h_t^{(1)} = 0.002 + 0.02\varepsilon_{t-1}^2 & Y_{t-1} \leq 0 \\ h_t^{(2)} = 0.005 + 0.04\varepsilon_{t-1}^2 & Y_{t-1} > 0 \end{cases}$$

We simulate 500 series with length n=200 and n=400 each, dropping the first 50 observations in order to eliminate the influence of initial seed, and compute the MLE estimates running the IWLS

(Iterative Weighted Leas Squares, Mak and Li, 1994) algorithm implemented in Fortran.

Parameters are chosen to ensure the regularity conditions and to consider different degrees of dependence in each regime. Due to the complexity of the estimation algorithm the orders p an q are set to one in order to keep reasonable the number of parameters. The threshold values are set to zero both for the level and for the residuals in order to guarantee the twofold subdivision in subregimes in the simulation processes.

For each Monte Carlo replicate we apply the subseries approach considering respectively, for n=200,400, an $l_{(200)}=l_1+l_2=52$ and an $l_{(400)}=l_1+l_2=104$ with 296 and 148 subseries replicates for a total of 148.000 and 74.000 IWLS estimates. The results for n=200 and n=400 have not substantial difference, hence, for brevity, we report only the n=400 case.

Table 1 displays the principal descriptive statistics of the replicate histogram of the sampling distributions for the AR and the ARCH coefficients with n=400 and $l(n)$=104, figure 1 shows the graphical results of the kernel smoothed approximation.

Table 1: Descriptive Statistics of the replicate histograms (n=400)

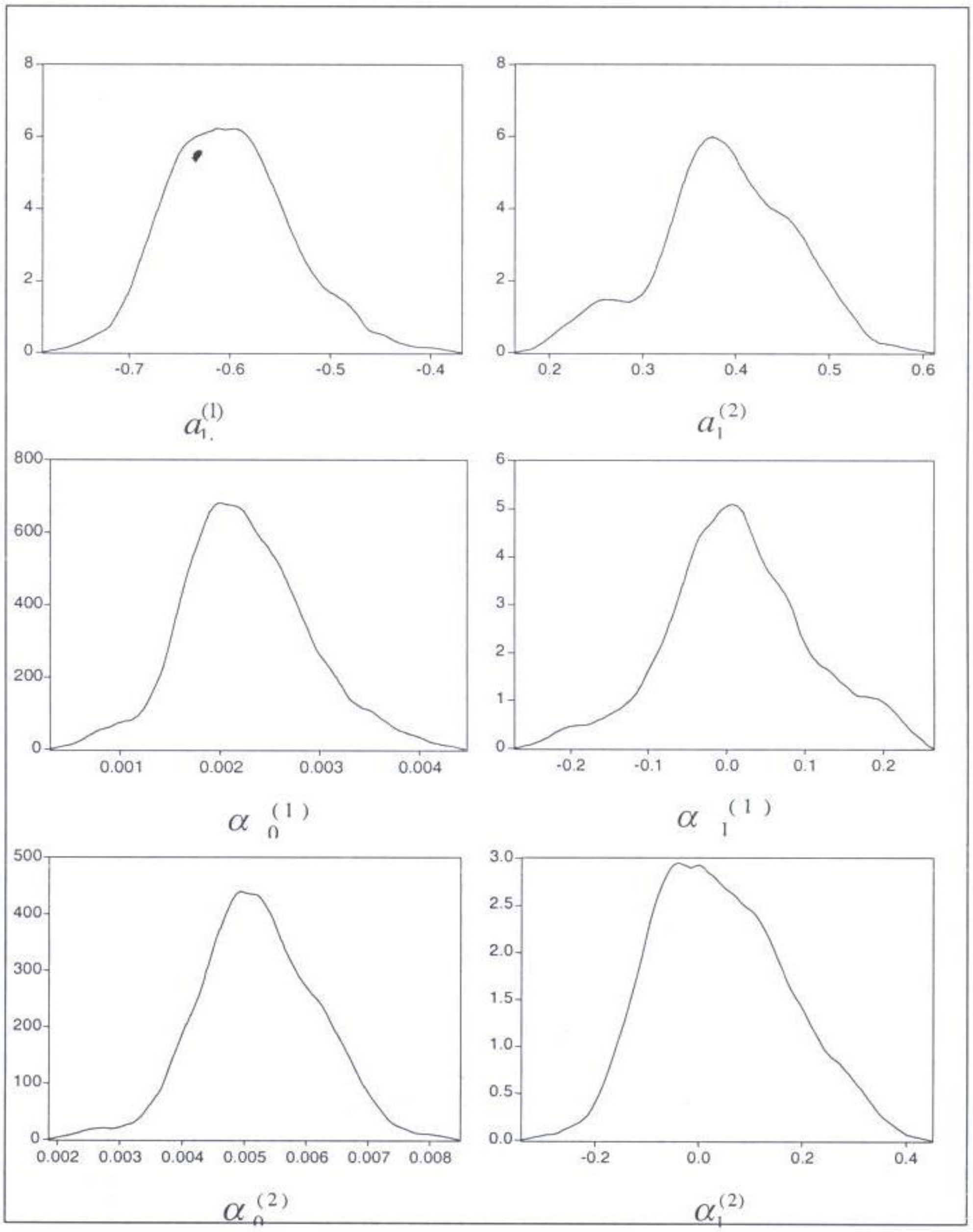
	$a_1^{(1)}$	$a_1^{(2)}$	$\alpha_0^{(1)}$	$\alpha_1^{(1)}$	$\alpha_0^{(2)}$	$\alpha_1^{(2)}$
Mean	-0.616	0.387	0.002	0.017	0.005	0.042
St.dev.	0.059	0.071	0.001	0.092	0.001	0.124
Skewness	0.420	-0.225	0.325	0.315	0.009	0.323
Kurtosis	3.127	2.845	3.323	0.3594	3.210	2.557

From the characteristic values and the graphical representations, the replicate histogram does seem compatible with a normal sampling distribution as confirmed by the Jarque-Bera normality tests (Tab.2).

Table 2: Jarque-Bera Normality Test

	$a_1^{(1)}$	$a_1^{(2)}$	$\alpha_0^{(1)}$	$\alpha_1^{(1)}$	$\alpha_0^{(2)}$	$\alpha_1^{(2)}$
J-B	3.011	0.933	2.164	0.058	0.1864	2.556
p-value	0.221	0.626	0.338	0.971	0.910	2.557

Figure 1*: Smoothed replicate histograms for n=400 and l(n)=104*



In order to numerically evaluate the effectiveness of the procedure in terms of bias reduction we compare the average value of the IWLS estimates computed on each Monte Carlo replicate with the same value calculated with the subseries approach.
As we can easily observe from the results of table 3, the SA value is less biased than the simple IWLS and the improvements are more evident for the coefficients of the ARCH component where the estimation problems are highly complex.

Table 3: Real value, IWLS and SA estimates

	$a_1^{(1)}$	$a_1^{(2)}$	$\alpha_0^{(1)}$	$\alpha_1^{(1)}$	$\alpha_0^{(2)}$	$\alpha_1^{(2)}$
θ	-0.600	0.400	0.002	0.020	0.005	0.040
$\hat{\theta}_{IWLS}$	-0.567	0.353	0.0016	0.013	0.0043	0.036
$\hat{\theta}_{SA}$	-0.616	0.387	0.002	0.017	0.005	0.042

5. Concluding remarks

The (double) threshold autoregressive conditional heteroscedastic model can be an useful tool for modelling time series with highly complex behaviour such as the case of financial or hydrological time series. However the lack of inference results for the parameters does not consent to utilise this structure in solving fitting and prediction problems.
The proposed subsampling procedure allows to obtain an approximation of the parameters estimators sampling distributions and to reduce the bias of the IWLS estimates where the classical asymptotic theory can't be easily applied. The suggested choice of the subseries length ensure to preserve the original dependence structure. The complexity of the estimation algorithm and the elevate number of iteration for each series make the estimate procedure with the SA not so fast in CPU time. Nevertheless it still remain an efficient method to obtain diagnostic information about the shape of sampling distributions and to approach inference problems in complex-non-linear structure.

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