Quaderni di Statistica
Vol. 4, 2002

On first-passage problem for a non-singular Gaussian discrete-time series

Elvira Di Nardo
Dipartimento di Matematica, Università degli Studi della Basilicata
E-mail: dinardo@unibas.it

Summary: For a non-singular Gaussian discrete-time series, the evaluation of the first passage time probability distribution through a bounded time-varying boundary is considered by using orthant probabilities. Sufficient conditions are given for the existence of finite mean and variance of the first passage time random variable. The computation of orthant probabilities is examined regard to the existing literature. As working examples, the first passage time problem for the discrete-time versions of the fractional white noise is considered.

Key words: First passage time, Gaussian discrete-time series, Orthant probability, Fractional white noise.

1. Introduction

In the field of stochastic processes, special care has been put on the discussion of the so-called first passage time (FPT) problem. This consists in the evaluation of the probability distribution of the time when first the process modeling the system's dynamics enters a preassigned critical region or reaches some preassigned boundary, for the general setting see Redner (2001). The main results on FPT problems are mainly focused on stochastic processes of diffusion type, where the Markov property plays a leading role in handling the related transition probability density function (pdf), see Ricciardi *et al.* (1999) for a review. However the knowledge of the free transition pdf is often not sufficient to determine the transient functions in the presence of absorbing time dependent boundary. So, apart from few special cases, no closed form of the FPT pdf are available in the literature and numerical algorithms or simulation procedures have been resorted in order to get more information on the features of the FPT pdf. Even for the diffusion processes some numerical methods are available in order to estimate the FPT pdf with boundaries having different shapes (see for example Di Nardo *et al.*, 2001 and Giraudo *et al.*, 2001). If in addition the Markov property is pulled down, the available analytical results on the FPT pdf are scarse and hard to manage for practical purposes so that the simulation procedures are the only beaten way (Di Nardo *et al.*, 2000).

When the time is discrete, many results on FPT problems have been achieved concerning the random walks or more generally the Markov chains (Redner, 2001). Kedem (1980) has proposed an interesting approach by clipping the time series in a binary one depending on the crossing or not some specified constant level. The main results are focused on binary series described by first-order or second-order stationary Markov chains in statistical equilibrium. However as in the case of stochastic processes, if the Markov property is relaxed, the results on FPT problems become poor and fragmentary.

The aim of this paper is to analysed the FPT problem for a non-singular Gaussian discrete-time series. In Section 2 the mathematical background for the FPT problem is first formulated. In particular, it is showed how the FPT p robability distribution is strictly correlated to the so-called orthant probabilities: this connection suggests a new way for the numerical evaluation of the orthant probabilities through an hoc simulation procedure. In Section 3 a short review is given concerning the numerical methods and the Monte-Carlo ones available in the literature for the computation of such orthant probabilities. In Section 4, as working examples, we will analyse the FPT problem for the discrete-time versions of the fractional Gaussian white noise (fGwn). Some concluding remarks are given in closing.

2. A first passage time analysis

Define the *FPT random variable* (r.v.) of a discrete-time series X_t through a boundary S_t as

$$T = \min_{t \in N} \{t : X_t \geq S_t\}, \tag{1}$$

where $P(X_0 \leq S_0) = 1$. In the following, suppose $P(X_0 = x_0) = 1$ with $x_0 < S_0$ and in order to simplify set $x_0 = 0$.

The aim of this section is to characterize the probability distribution of T when X_t is a non-singular Gaussian discrete-time series such that $E[X_t] = 0$. Remark that the hypothesis on the zero means could be relaxed.

In order to evaluate the probability distribution of T, first observe that

$$P(T = k) = P(X_1 < S_1, X_2 < S_2, \ldots, X_{k-1} < S_{k-1}, X_k \geq S_k), \tag{2}$$

and due to the settings on X_t, it is

$$P(T = k) = \int_{D_k} \frac{1}{(2\pi)^{k/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} Q_k(\mathbf{x}; \mathbf{0}, \Sigma)\right\} d\mathbf{x}$$

where

$$D_k = \{\mathbf{x} \equiv (x_1, x_2, \ldots, x_k) \in \mathbf{R}^k : \ x_i < S_i \ i = 1, 2, ..., k-1, \ x_k \geq S_k\}$$

and

$$Q_k(\mathbf{x}; \mathbf{0}, \Sigma) = \mathbf{x}^T \Sigma^{-1} \mathbf{x}$$

with $(\Sigma)_{ij} = E[X_i X_j] = \rho_{ij}$ the variance-covariance matrix. Equation (2) could be rewritten as

$$P(T = k) = \begin{cases} 1 - P(X_1 < S_1) & \text{if } k = 1 \\ P\left(\cap_{i=1}^{k-1} \{X_i < S_i\}\right) - P\left(\cap_{i=1}^{k} \{X_i < S_i\}\right), & \text{if } k > 1. \end{cases} \tag{3}$$

Usually, the rectangular regions

$$O_k = \{\mathbf{x} \equiv (x_1, x_2, \ldots, x_k) \in \mathbf{R}^k : \ x_i < S_i \ i = 1, 2, ..., k\}$$

are named *orthants* regions: that is why the probabilities appearing in (3)

$$\begin{aligned} P_k(\mathbf{S}, \Sigma) &= P\left(\cap_{i=1}^k \{X_i < S_i\}\right) \\ &= \int_{O_k} \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} Q_k(\mathbf{x}; \mathbf{0}, \Sigma)\right\} d\mathbf{x} \end{aligned} \tag{4}$$

where $\mathbf{S} \equiv (S_1, S_2, \ldots, S_k)$ for any integer k, are named *orthant probabilities* in the literature.

Then the problem of characterize (2) could be bring back to the evaluation of the orthant probabilities (4) for a multivariate normal distribution. Unless $k \leq 3$ or $\Sigma = I$ (Thisted, 1988), no closed-form expressions are known for $P_k(\mathbf{S}, \Sigma)$, so the evaluation of (4) leads to the numerical methods as it will be shown in details in the next section.

Let us give some properties on the orthant probabilities (4). First, we recall a result due to Plackett (1954) and later stated by Slepian (1962) in a more general form.

Theorem 1. (Plackett)

Let $\mathbf{X} \equiv (X_1, X_2, \ldots, X_n)'$ *be a multivariate normal r.v. with* $(\Sigma_X)_{ij} = \rho_{ij}^X$ *and* $\mathbf{Y} \equiv (Y_1, Y_2, \ldots, Y_n)'$ *be a multivariate normal r.v. with* $(\Sigma_Y)_{ij} = \rho_{ij}^Y$. *Let* $\rho_{ii}^X = \rho_{ii}^Y = 1$, *for* $i = 1, 2, \ldots, n$. *If* $\rho_{ij}^X \geq \rho_{ij}^Y$, $i, j = 1, 2, \ldots, n$ *then*

$$P_k(\mathbf{S}, \Sigma_X) \geq P_k(\mathbf{S}, \Sigma_Y), \quad k = 1, 2, \ldots, n. \tag{5}$$

The previous theorem states that $P_k(\mathbf{S}, \Sigma)$ is a function strictly increasing of $\rho_{ij} \in I_{(i,j)} = \{\rho_{ij} : \Sigma \text{ is positive definite}\}$. Using this result, we claim some bounds on the orthant probabilities (4).

Theorem 2.

For a non-singular Gaussian discrete-time series X_t *with zero means and* $\rho_{ii} = 1, i \in \mathbf{N}$ *and for the boundary* S_t *set*

$$\rho_{\max} = \max_{i \neq j} \rho_{ij},\ \rho_{\min} = \min_{i \neq j} \rho_{ij},\ S_{\max} = \max_{t \geq 0} S_t,\ S_{\min} = \min_{t \geq 0} S_t. \tag{6}$$

If $\rho_{\max}, \rho_{\min} \in [0, 1)$ *it is*

$$B_k(S_{\min}, \rho_{\min}) \leq P_k(\mathbf{S}, \Sigma) \leq B_k(S_{\max}, \rho_{\max}) \tag{7}$$

where $P_k(\mathbf{S}, \Sigma)$ is given in (4) and

$$B_k(S, \rho) = \int_{-\infty}^{\infty} \Phi^k \left(\frac{S + \sqrt{\rho} z}{\sqrt{1-\rho}} \right) \phi(z) dz, \quad S \in \mathbf{R},\ \rho \in [0, 1) \tag{8}$$

with

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\}, \quad \Phi(x) = \int_{-\infty}^{x} \phi(z) dz, \quad x \in \mathbf{R}. \tag{9}$$

Proof

Using the Plackett result (5), it is

$$P_k(\mathbf{S}, \Sigma) \leq P_k(\mathbf{S}, \Sigma_{\rho_{\max}}) \leq P_k(\mathbf{S}_{\max}, \Sigma_{\rho_{\max}})$$

where $\Sigma_{\rho_{\max}}$ is the variance-covariance matrix with 1 over the principal diagonal and $\rho_{\max}$ elsewhere and $\mathbf{S}_{\max}$ is a vector whose k components are equal to $S_{\max}$. The random vector with normal distribution function $N(\mathbf{0}, \Sigma_{\rho_{\max}})$ is built with exchangeable r.v.'s and so (Tong, 1990)

$$P_k(\mathbf{S}_{\max}, \Sigma_{\rho_{\max}}) = B_k(S_{\max}, \rho_{\max})$$

with $B_k(S_{\max}, \rho_{\max})$ given in (8). This proves the second inequality in (7), the first inequality following by similar arguments.

Remark 1.

If $\rho_{max} \leq 0$ it is still possible to find un upper bound for (4) by using again the Plackett inequality (5)

$$P_k(\mathbf{S}, \Sigma) \leq P_k(\mathbf{S}, I) \leq P_k(\mathbf{S}_{\max}, I) = \Phi^k(S_{\max}) \tag{10}$$

where I is the identity matrix, $S_{\max}$ is defined in (6) and $\Phi(x)$ is given in (9).

If the boundary S_t is bounded, for k growing up the probability that the process X_t remains under the boundary goes to zero, that is

$$\lim_{k \to \infty} P_k(\mathbf{S}, \Sigma) = 0. \tag{11}$$

Indeed, if

$$S_{\max} = \max_{t \geq 0} S_t < \infty,$$

the asymptotic result (11) follows taking the limit for k to infinite in (7) if $\rho_{\max} \geq 0$ or in (10) if $\rho_{\max} \leq 0$.

This last result turns to be useful into testing if the FPT r.v. is fair, that is equivalent to verify if $P(T < \infty) = 1$.

Theorem 3. *If the boundary S_t is bounded, then the FPT r.v. is fair.*
Proof

Indeed, by (2) and (3), it is

$$\begin{aligned} P(T \leq k) &= 1 - P_1(\mathbf{S}, \Sigma) + \sum_{j=2}^{k} [P_{j-1}(\mathbf{S}, \Sigma) - P_j(\mathbf{S}, \Sigma)] \\ &= 1 - P_k(\mathbf{S}, \Sigma). \end{aligned}$$

Then, taking the limit as k going to infinite and recalling (11), it results $P(T < \infty) = 1$.

About the FPT mean, let state the following proposition.

Proposition 1. *If the boundary S_t is bounded, then*

$$E[T] = 1 + \sum_{k=1}^{\infty} P_k(\mathbf{S}, \Sigma). \tag{12}$$

Proof

It is

$$E[T] = 1 - P_1(\mathbf{S}, \Sigma) + \sum_{k=2}^{\infty} k\, [P_{k-1}(\mathbf{S}, \Sigma) - P_k(\mathbf{S}, \Sigma)]$$

and being for (7) or (10)

$$\lim_{k \to \infty} k P_k(\mathbf{S}, \Sigma) = 0,$$

equation (12) follows.

Proposition 2. *If the boundary S_t is bounded, then*

$$E[T^2] = 1 + \sum_{k=1}^{\infty} (2k+1) P_k(\mathbf{S}, \Sigma). \tag{13}$$

Proof

Being for (7) or (10)

$$\lim_{k\to\infty} k^2 P_k(\mathbf{S}, \Sigma) = 0,$$

it is

$$\begin{aligned} E[T^2] &= 1 - P_1(\mathbf{S}, \Sigma) + \sum_{k=2}^{\infty} k^2 \left[P_{k-1}(\mathbf{S}, \Sigma) - P_k(\mathbf{S}, \Sigma) \right] \\ &= 1 + \sum_{k=1}^{\infty} [(k+1)^2 - k^2] P_k(\mathbf{S}, \Sigma) \end{aligned}$$

by which equation (13) follows.

It is interesting to evaluate some bounds for $E[T]$, if there exist. Indeed, if $E[T] < \infty$ for some X_t, it is possible to evaluate numerically the series in (12) with a determined tolerance. The bounds for the FPT mean depend on the structure of the variance-covariance matrix Σ, as it will be shown in the following.

If $\rho_{\max} \leq 0$ then $E[T] < \infty$, being from (10)

$$E[T] \leq \frac{1}{1 - \Phi(S_{\max})} \tag{14}$$

where $S_{\max}$ and $\rho_{\max}$ are defined in (6) and $\Phi(x)$ is given in (9). By (13), it is

$$E[T^2] \leq \frac{1 + \Phi(S_{\max})}{[1 - \Phi(S_{\max})]^2}$$

so that $Var[T] < \infty$.

If $\rho_{\min} \geq 0$ and $\rho_{\max} < 1$ from (7) it is

$$A(S_{\min}, \rho_{\min}) \leq E[T] \leq A(S_{\max}, \rho_{\max}) \tag{15}$$

where

$$A(S, \rho) = \int_{-\infty}^{\infty} a(z, S, \rho) dz, \quad a(z, S, \rho) = \frac{\phi(z)}{1 - \Phi\left(\frac{S + \sqrt{\rho} z}{\sqrt{1-\rho}}\right)}, \tag{16}$$

with $S \in \mathbf{R}$, $\rho \in [0, 1)$. Remark that $A(S, \rho) \in \mathbf{R} \cup \{\infty\}$. Indeed, if $\rho \in [0, 0.5)$, the function $a(z, S, \rho)$ has finite values, $\lim_{z \to \pm\infty} a(z, S, \rho) = 0$ and so $A(S, \rho) < \infty$. However the numerical evaluation of (16) could produce some problems if S is too greater. On the contrary, if $\rho \in [0.5, 1)$, it is $\lim_{z \to \infty} a(z, S, \rho) = \infty$ likewise $\phi(z)/(1 - \Phi(z))$ so that the integral in (16) is infinite.

3. The numerical evaluation of the FPT probability distribution

The problem of approximating the orthant probabilities (4) for a multivariate normal distribution has a rich history (for a review see Tong, 1990) not only within numerical analysis but also owing to its manifold applications as symmetric order statistics (Horn, 1982), as investigation on the number of peaks in a stationary Gaussian process (Ku and Seneta, 1994), as fluctuation of sums of r.v.'s (Nesch, 1993). Most earliest results concern asymptotic expansions of the multivariate normal density function so that (4) could be express as infinite series and then to approximate. This method has been used especially for $k = 2$ and $k = 3$ (cf. for example Gupta, 1963 and more recently Vasicek, 1998) although more competitive algorithms have been recently proposed (see Genz, 2001 and references therein). A completely different approach to the evaluation of (4) is the Plackett's formula (Plackett, 1954) consisting in a dimension-reduction formula based on an hoc conditioning. Other methods lie in available numerical integration software, as Monte-Carlo methods or subregion adaptive numerical methods. However, the infinite integration limits need to be carefully handled either by using some type of transformations into a finite region (see for example Genz, 1992 and 1993 and references therein) either using selected cutoff values.

In the special case $\mathbf{S} \equiv \mathbf{0}$ the problem of interest has been to find expressions of (4) in terms of the correlation coefficients. In this case, solutions are available in closed forms for $k = 2$ and $k = 3$ while for $k \geq 4$ integral representations are obtained in some special cases. Some analytical decompositions of (4) into a combination of several multiple integrals of low order are given in Sun (1988), see references therein for different kind of decomposition formulae. A n-dimensional recursion formula is found in Zhongren and Kedem (1999) by means of a linear transformation followed by a polar coordinates transformation.

At last other methods approximating (4) depend on the structure of Σ : for example when Σ has a tridiagonal form or other factorial structures with a small number of factors (for similar matters see Tong, 1990 and Thisted, 1988).

At the moment, with up to $k = 10$, there are available different methods by which orthant probabilities can be robustly and reliably computed at low to moderate accuracy levels. High accuracy or high dimension problems can require long computation times and it is still not clear what is the best method for this type of problem.

This is why, in the following, three methods are proposed whose results and comparisons are displayed in the next section.

3.1 The NAG software library

The NAG software library (available at http://www.nag.co.uk/) includes the soubroutine G01HBF written in FORTRAN 77 in order to evaluate (4) for $k \leq 10$.

The advantage of this routine is the chance to establish the relative accuracy tol of the computation on entry.

The probability distribution (4) is evaluated as the product of the conditioned probability of $X_1, X_2, \ldots, X_{n-2}$ given X_{n-1} and X_n and the marginal bivariate normal distribution of X_{n-1} and X_n. The bivariate normal probability is evaluated by the routine G01HAF based on the method described by Divgi (1979). For the left over integral of $n-2$ dimension, a numerical adaptive quadrature is used and in case of some infinite limits the cut-off point region is given by $\Phi^{(-1)}[tol/(10n)]$, where $\Phi^{(-1)}$ is the inverse univariate normal distribution function.

3.2 The Geweke-Hajivassiliou-Keane simulator

The Geweke-Hajivassiliou-Keane (GHK) simulator is a Monte-Carlo method, first proposed by Geweke (1989). It is based on sampling from recursive truncated normal r.v.'s after a Choleski transformation. In a review paper, Hajivassiliou *et al.* (1996) analyse the properties of a number of available simulators of (4) and find that the GHK one performs all other methods by keeping a good balance between accuracy and computational costs.

The idea is the following. Let $\{Y_i\}_{i\in\mathbf{N}}$ be a sequence of i.i.d. r.v.'s with standard normal distribution. Set the events

$$A_i = \left\{-\infty < Y_i < \frac{S_i - \sum_{j=1}^{i-1} c_{ji} Y_j}{c_{ii}}\right\}$$

with c_{ij} elements of the lower triangular Cholesky decomposition of the variance-covariance matrix Σ. Then

$$P_k(\mathbf{S},\Sigma) = P(A_1)P(A_2|A_1)\cdots P(A_k|A_1,A_2,\ldots,A_{k-1})$$

and its estimation could be written as

$$\tilde{P}_k(\mathbf{S},\Sigma) = \frac{1}{N}\sum_{n=1}^{N} P(A_1)P(A_2|y_{1,n})\cdots P(A_k|y_{1,n};y_{2,n};\ldots;y_{k-1,n}) \tag{17}$$

where $\{y_{i,n}\}$ are drawn sequentially from independent standard normal distributions truncated from the events A_i.

At http://econ.lse.ac.uk/staff/vassilis/ the routine for computing (17) up to $k=20$ is available in FORTRAN 77 and in GAUSS.

It has been proved that the estimator $\tilde{P}_k(\mathbf{S},\Sigma)$ is consistent in N; for a given N its accuracy declines with k, while increases with N for any k.

3.2 The Genz transformation

Genz (1993) proposes a sequence of three transformations in order to transform the original integral (4) into an integral over an unit hypercube

$$P_k(\mathbf{S}, \Sigma) = e_1 \int_0^1 e_2 \int_0^1 \dots \int_0^1 e_k \int_0^1 d\mathbf{w}, \tag{18}$$

with $d\mathbf{w} \equiv (dw_1, \dots, dw_k)$,

$$e_i = \Phi \left\{ \frac{S_i - \sum_{j=1}^{i-1} c_{ij} \Phi^{(-1)}(w_j e_j)}{c_{ii}} \right\}, \quad i = 1, 2, \cdots, k,$$

where c_{ij} are the elements of the lower triangular Cholesky decomposition of the variance-covariance matrix Σ, $\Phi(x)$ is given in (9) and $\Phi^{(-1)}(x)$ is its inverse. The overall transformation forces an ordering on the integration variables w_i that makes the evaluation of (18) more suitable for the subregion adaptive procedure suggested in Berntsen, Espelid and Genz (1991).

At http://www.sci.wsu.edu/math/faculty/genz/homepage the routine for computing (18) up to $k = 100$ is available in FORTRAN 77 and in GAUSS. It returns the estimated absolute error, with 99% confidence level. Some termination status parameters inform about an atypic completion.

4. Working example: the discrete-time fractional Gaussian white noise

Recent studies have shown how the fractional Brownian motion (fBm) and the fractional Gaussian white noise (fGwn) are useful in characterizing subsurface heterogeneities in addition to geophysical time series. But the first employment of the fGwn was within the hydrologic time series, due to the presence of long-range dependence in the data at bottom of the famous phenomenon now called the Hurst effect. Different models have been proposed in the literature aiming to the description of the statistical features exhibited by the hydrologic time series:

i) Mandelbrot and Van Ness (1968) define the discrete-time version of the fGwn

$$W_H(k) = \Delta B_H(k) = B_H(k) - B_H(k-1), \quad k = 1, 2, \ldots$$

where $\{B_H(t), t \geq 0, H \in (0,1]\}$ is the fBm, as well as the white Gaussian noise was constructed in order to give some meaning to the concept of derivative of the Brownian motion;

ii) Hosking (1981) constructs the ARFIMA (fractional autoregressive integrated moving average) models by applying traditional ARMA models to the fractional differences of the data.

The fGwn of Mandelbrot and Van Ness is incapable of fitting short-range dependence patterns in many hydrological records. Instead, the ARFIMA models are potentially powerful tool for modeling stationary hydrological records, since the fractional differencing allows to model the long-memory effects while the ARMA part provides the right flexibility in modeling the short-memory pattern.

Within the hydrologic time series, the analysis of extreme values could be very useful in predicting some exceptional circumstances (as overflows or floods): the classical approach would provide the study of the maximum of the series over a fixed time interval, assuming the independence of the maxima over contiguous intervals and so losing largely information. An alternating approach consists in the analysis of the crossing of a prefixed level and takes into account the correlation structure underlying the series.

Let us point out that the number of level crossing has many other applications, different from the forecasting of extreme values or periodicity in hydrologic time series. For example, the zero crossing problem is strictly related to the estimation of the spectral density from a given time series. This because frequent axis-crossing blows up high frequencies while fewer axis-crossing shifts this emphasis towards low frequencies. Kedem and Slud (1981) proposes a goodness of fit test for time series models as application of higher order zero-crossings.

4.1 The Mandelbrot and Wallis fractional white noise

The fBm $B_H(t)$ belongs to the class of self-similar processes characterized by distributions invariant to a scale-change (for a review of self-similar processes see Carbon, 1991). Let us recall that the fBm has stationary, but non independent increments, except for $H = 0.5$, and it is the only Gaussian self-similar process with stationary increments. The FPT problem for the fBm process has been recently analyzed by Michna, (1999).

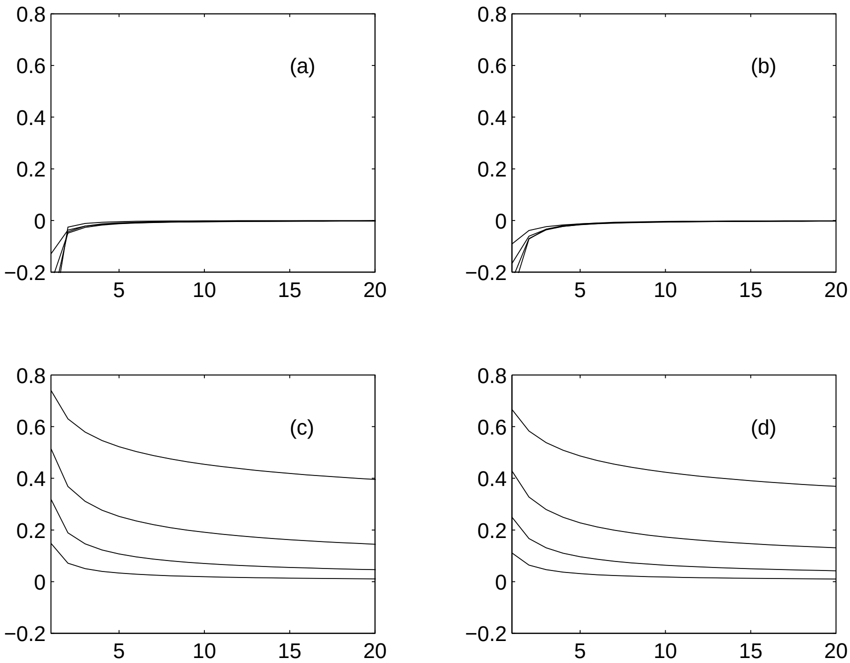

Figure 1. Plots of $\gamma_{W_H}(k)$ for $H = 0.1, 0.2, 0.3, 0.4$ in (a) (bottom to top) and for $H = 0.6, 0.7, 0.8, 0.9$ in (c) (bottom to top). Plots of $\gamma_{AR}(k)$ for $d = -0.1, -0.2, -0.3, -0.4$ in (b) (bottom to top) and for $d = 0.1, 0.2, 0.3, 0.4$ in (d) (bottom to top).

The discrete-time series fGwn $\{W_H(k)\}_{k\in\mathbf{N}}$ is Gaussian, stationary, with zero means and correlation function

$$\gamma_{W_H}(k) := \frac{1}{2}\left\{|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right\}. \tag{19}$$

Plots of $\gamma_{W_H}(k)$ for different choices of H are showed in Figure 1(a)-(c), by which it is clear as the parameter H characterizes the behaviour of the fGwn: if $H \in (0, 0.5)$, the r.v.'s $W_H(k)$ have serial negative correlations which decay rapidly and sum to zero: such a time series often exhibits sample paths where high and low values tend to consecutively alternate, a property known as anti-persistent; if $H = 0.5$, the r.v.'s $W_H(k)$ are incorrelated and they represent the discrete version of the classical white noise; if $H \in (0.5, 1)$ the r.v.'s $W_H(k)$ display a long-memory behavior since their positive serial correlation slowly decays.

Table 1. Evaluations of the FPT probability distribution

k	NAG	GHK	GENZ
1	0.158655253931457	0.158655253931457	0.158655253931457
2	0.149891261355585	0.147247930548936	0.149891262180832
3	0.127537741153479	0.130466070445761	0.127518359911067
4	0.105700370137505	0.105053570150835	0.105736185260652
5	0.087230732128142	0.085007711653340	0.086695793973868
6	0.070597972846620	0.071552935565272	0.070872214353787
7	0.057233171605209	0.058618865253761	0.057819898428268
8	0.048108661918773	0.048790051405705	0.046719854696929
9	0.036973338392874	0.037492871980221	0.038057818797014
10	0.030732400618266	0.030997144621598	0.030787302699045
11		0.023230924405503	0.024661193822441
12		0.020077881818070	0.020037105897441
13		0.016389751261822	0.016168997029792
14		0.012899636264356	0.012934558801507
15		0.010390027152509	0.010536683471073
16		0.008302585043037	0.008410738086619
17		0.006735097099198	0.006730555664805
18		0.005401426732562	0.005497252577249

Being $\gamma_{W_H}(0) = 1$ and $\gamma_{W_H}(k) < 1$ for any integer k, the variance-covariance matrix Σ_{W_H} is positive definite: the fGwn $W_H(k)$ is non-singular and its FPT r.v. through a bounded boundary is fair. The FPT mean is finite. Indeed, if $H \in (0, 0.5)$ it is $\rho_{\max} = \max_k \gamma_{W_H}(k) \leq 0$ so that (14) holds. If $H \in [0.5, 1)$, being $\rho_{\min} = \min_{k>0} \gamma_{W_H}(k) = 0$ and $\rho_{\max} = 2^{2H-1} - 1 > 0$ it is

$$\frac{1}{1 - \Phi(S_{\min})} < E[T] < A(S_{\max}, \rho_{\max})$$

with $A(S_{\max}, \rho_{\max})$ in (16), the inequality being strictly due to the decreasing of (19).

In Table 1, evaluations of the FPT probability distribution by using the NAG routine with $tol = 0.01$, the GHK simulator with $N = 500$ and the Genz trasformation with $H = 0.2$ and $S = 1$ are displayed: observe that the values obtained from the NAG routine and the Genz method are comparable, while for those obtained from the GHK simulator it is necessary to increase N in order to have a better accuracy.

4.2 The ARFIMA$(0, d, 0)$ model

The ARFIMA$(0, d, 0)$ time series is a different discrete-time version of the fGwn, built by using a symbolic generalization of the first difference operator $\nabla = 1 - B$, where B is the backward-shift operator $BX_t = X_{t-1}$ (Hosking, 1981). Such a generalization is the fractional difference operator

$$\nabla^d = (1 - B)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-B)^k.$$

Recalling that a sequence of standard Gaussian i.i.d.r.v.'s $\{a_k\}$ is a discrete version of the Gaussian white noise, the discrete-time version of the fGwn is constructed taking the $(1/2 - H)$th fractional difference of $\{a_k\}$, that is $X_t = \nabla^{1/2-H} a_t$. Setting $d = H - 1/2$, the ARFIMA$(0, d, 0)$ time series satisfies

$$\nabla^d X_t = a_t.$$

When $H \in (0,1)$, it is $d \in (-0.5, 0.5)$, and it could be proved that the ARFIMA$(0,d,0)$ process is Gaussian, stationary with zero means and correlation function

$$\gamma_{AR}(k) = \frac{d(d+1)\cdots(d+k-1)}{(1-d)(2-d)\cdots(k-d)}. \tag{20}$$

As before, if $d \in (0, 0.5)$, the ARFIMA$(0,d,0)$ model displays a long-memory behavior since its positive serial correlation decays hyperbolically to zero, see Figure 1(d); if $d = 0$, this model is again the discrete version of the Gaussian white noise; if $d \in (-0.5, 0)$ the model has negative serial correlation which decays rapidly, so there is a short memory and an anti-persistent behaviour, see Figure 1(b). Following the same arguments given in the previous subsection, the ARFIMA$(0,d,0)$ process is non-singular with the FPT probability distribution given by (3) and finite FPT mean.

It is natural to compare the two time series that describe a discrete-time version of the fGwn: the ARFIMA$(0,d,0)$ process and the $W_H(k)$ one. Even if the structure of the related correlation functions is different, they become similar to k^{2d-1} for large k. In other words the two discrete-time versions of the fGwn "looks-like" similar.

This similarity is reflected also within the FPT probability distribution. In Figure 2, plots of the FPT probability distribution are reported for the ARFIMA$(0,d,0)$ model and for the $\{W_H(k)\}$ time series. The Figure 2(a) is related to the ARFIMA$(0,0.2,0)$ that corresponds to $\{W_{0.7}(k)\}$ in Figure 2(c): the two plots are very similar. In particular, it is $P(T_{AR} \leq 50) = 0.9942$ whereas $P(T_W \leq 50) = 0.9924$, having set with T_{AR} the FPT r.v. for the ARFIMA model and with T_W the FPT r.v. for the $\{W_H(k)\}$ series. The estimated FPT mean is about 8.3551 in the first case and 8.7717 in the second one. If the boundary is moved away, the FPT probability distribution decreases (compare Figure 2(a) and Figure 2(d)). The short memory behavior of the ARFIMA$(0,d,0)$ model reflects upon the Figure 2(b) where $d = -0.2$: being $P(T_{AR} \leq 50) = 0.9999$ the FPT occurs before respect to the case $d = 0.2$ as well as the estimated FPT mean 5.6811 is less. So, as d increases, the FPT occurs later.

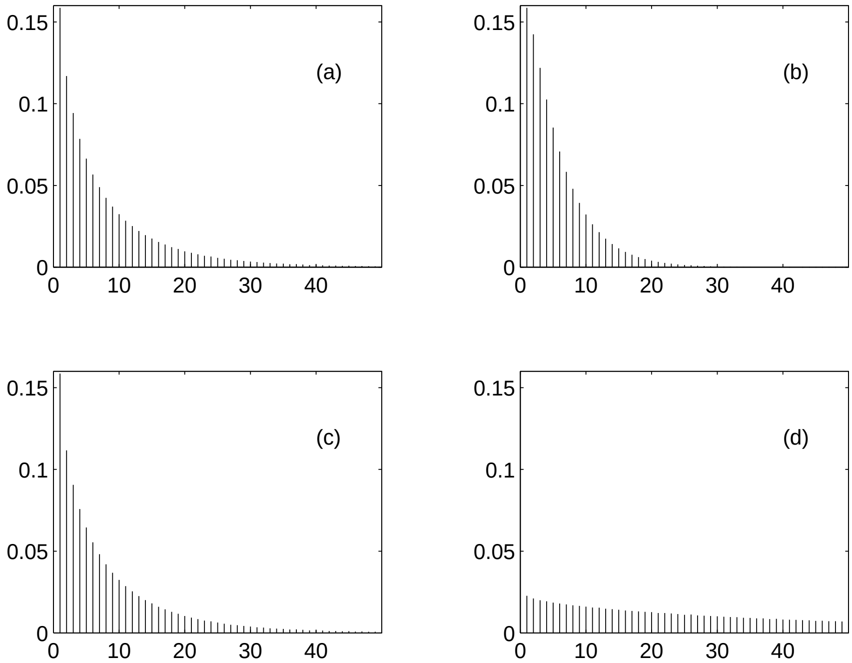

Figure 2. Plots of the FPT probability distributions for the ARFIMA$(0, d, 0)$ model with $d = 0.2$ and $S = 1$ in (a), with $d = -0.2$ and $S = 1$ in (b), with $d = 0.2$ and $S = 2$ in (d) and for the $\{W_H(k)\}$ time series with $H = 0.7$ and $S = 1$ in (c).

5. Conclusions and further developments

In the present paper, it has been shown the existent relation between the orthant probabilities and the FPT problem for a non-singular Gaussian discrete-time series so that it will be possible to study how the geometry of the absorbing boundary influences the shape of the FPT probability distribution. Applications in the forecasting of extreme values for the ARFIMA models or in the detection and estimation of discrete spectral components would be object of further investigations. What is more, the underlined relation between the orthant probabilities and the FPT problem suggests a new Monte-Carlo method in order to estimate the orthant probabilities differently from the simulators described in Section 3: a way that will be covered in a short while. Indeed, if the FPT r.v. is fair, it would be convenient to simulate the sample paths of the process, to record their first crossing instants and then to estimate (4) by its relative frequency.

References

Berntsen J., Espelid T.O., Genz A. (1991) An Adaptive Multidimensional Integration Routine for a Vector of Integrals, *Association for Computing Machinery. Transactions on Mathematical Software* , 17, 452–456.

Carbon M. (1991) Les Processus Self-Similaires, *Publications de l'Institut de Statistique de l'Université de Paris*, 36, 43–70.

Di Nardo E., Nobile A.G., Pirozzi E., Ricciardi L.M. (2001) A Computational Approach to First-Passage-Time Problem for Gauss-Markov Processes, *Advances in Applied Probability*, 33, 453–482.

Di Nardo E., Nobile A.G., Pirozzi E., Ricciardi L.M., Rinaldi S. (2000) Simulation of Gaussian Processes and First Passage Time Densities Evaluation, *Lecture Notes in Computer Science*, 1798, 319–333.

Divgi D.R. (1979) Calculation of Univariate and Bivariate Normal Probability Functions, *The Annals of Statistics*, 7, 903–910.

Genz A. (1992) Numerical Computation of Multivariate Normal Probabilities, *Journal of Computational and Graphical Statistics*, 1, 141–149.

Genz A. (1993) Comparison of Methods for the Computation of Multivariate Normal Probabilities, *Computing Science and Statistics*, 25, 400–405.

Genz A. (2001) Numerical Computation of Bivariate and Trivariate Normal Probabilities, Preprint.

Geweke J. (1989) Efficient Simulation from the Multivariate Normal Distribution Subject to Linear Inequality Constraints and the Evaluation of Constraint Probabilities, Mimeo, Duke University.

Giraudo M.T., Sacerdote L., Zucca C. (2001) A Monte-Carlo Method for the Simulation of First Passage Times of Diffusion Processes, *Methodology and Computing in Applied Probability*, 3, 215–231.

Gupta S.S. (1963) Probability Integrals of Multivariate Normal and Multivariate t, *The Annals of mathematical statistics*, 34, 792–828.

Hajivassiliou V., McFadden D., Ruud P. (1996) Simulation of Multivariate Normal Rectangle Probabilities and their Derivatives. Theoretical and Computational results, *Jornal of Economy*, 72 (1-2), 85–134.

Horn P. S. (1982) The Finite Distribution of t Statistics Based on Two Symmetric Order Statistics as a Sum of Orthant Probabilities, *Biometrika*, 69, 681–682.

Hosking J.R.M. (1981) Fractional Differencing, *Biometrika*, 68-(1), 165–176.

Kedem, B. (1980) Binary time series. *Lecture Notes in Pure and Applied Mathematics*, 52. Marcel Dekker, Inc., New York.

Kedem B., Slud E. (1981) On goodness of fit of time series models: an application of higher order crossings. *Biometrika*, 68, 551–556.

Ku S., Seneta E. (1994) The Number of Peaks in a Stationary Sample and Orthant Probabilities, *Jornal of Time Series Analysis*, 15, 385–403.

Mandelbrot B.B., Van Ness J.W. (1968) Fractional Brownian Motions. Fractional Noises and Applications, *SIAM Review*, 10, 422–437.

Mandelbrot B.B., Wallis J.R. (1969) Computer Experiments with Fractional Gaussian Noises, Part 1, Averages and Variances, *Water Resources Research*, 5, 228–241.

Michna Z. (1999) On Tail probabilities and First Passage Times for Fractional Brownian Motion, *Mathematical Methods of Operations Research*, 49, 335–354.

Nesch P.E. (1993) A Multivariate Look at E. Sparre Andersen's Equivalence Principle, *Contemporary Mathematics*, 149, 331–337.

Plackett R.L. (1954) A Reduction Formula for Normal Multivariate Integrals, *Biometrika*, 41, 351–360.

Redner S. (2001) *A guide to First-Passage Processes*, Cambridge University Press, Cambridge.

Ricciardi L.M., Di Crescenzo A., Giorno V., Nobile A.G. (1999) An Outline of Theoretical and Algorithmic Approaches to First Passage Time Problems with Applications to Biological Modeling, *Mathematica Japonica*, 50, 247–322.

Sun H. J. (1988) A General Reduction Method for n-Variate Normal

Orthant Probability, *Communications in Statistics. Theory and Methods*, 17, 3913–3921.

Thisted R.A. (1988) *Elements of Statistical Computing: Numerical Computation*, Chapman and Hall, New-York.

Tong Y.L. (1990) *The Multivariate Normal Distribution*, Springer-Verlag, New-York.

Zhongren N., Kedem B. (1999) On Normal Orthant Probabilities, *Chinese Journal of Applied Probability and Statistics*, 15, 262–275.

Vasicek O.A. (1998) A Series Expansion for the Bivariate Normal Integral, *Journal of Computational Finance*, 1, 1–10.