Quaderni di Statistica
Vol. 10, 2008

A new R-ordering procedure to rank multivariate performances

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Summary: Performance analysis has become a strategic activity in managing complex systems. The aim of this activity is often benchmarking, that is the comparison among various units on the basis of perfomance. In statistical terms this corresponds to search for an ordering of units described by a set of p indicators. In this paper we propose a new reduced ordering procedure for multivariate observations based on determining a meaningful direction for the problem in the Euclidean p-dimensional space. Namely, the direction is the one which goes from the "worst" performing units to the "best", that is the "worst-best" direction. Some proposals for the "worst" and "best" case construction are offered along with the related reduced ordering procedure. Some geometric issues and statistical properties of the proposal are discussed.

Keywords: Reduced ordering, Archetypal analysis, Convex hull.

1. Introduction

Nowadays considered a strategic activity for any organization, performance analysis requires a system of indicators to measure inputs and outputs of ongoing processes, and to synthesize the outcomes (Perrin, 1998).

One of the aims of performance analysis is to compare units in order to search for best practices, thereby leading to improvements. This process usually ends up in a benchmarking activity, i.e. looking for best practices and best performances on the one hand, and worst practices on the other (Camp, 1989; Chap.2). Benchmarking is mainly based on a preliminary ranking of the units. This goal arises not only in business, but is widely found in many other fields, such as those related to a country's performance with regard to society, economy, environment and health (Saisana, 2004).

Statistically speaking, ranking is achieved through an ordering of the observations described by a set of p indicators, i.e. it entails ordering multivariate observations in the p-dimensional Euclidean space E^p. However, due to lack of natural ordering in E^p, this task cannot be pursued by simply extending univariate order concepts. Many approaches to multivariate ordering have been proposed in the literature (Tukey, 1975; Barnett, 1976 and discussion therein; Friedman and Rafsky, 1979; Eddy, 1985; Korhonen and Siljamäki, 1998; Liu *et al.*, 1999, Atkinson and Riani, 2000; Gentle, 2002).

However, most of these approaches are not well suited to ordering units measured through multivariate performance indicators, as they do not take into account the nature of data and the specific goals of benchmarking. In this paper we propose an ordering procedure based on the construction in the E^p space of a direction meaningful for benchmarking and for comparison among performers. Such a direction is built to coincide with that going from the "worst" performer to the performance driver - the "worst-best" direction.

The paper is organized as follows. In Section 2 we discuss the notion and the proposals for multivariate ordering. Section 3 provides our proposal to construct units such that they are intrinsically "worst" and "best", and the steps of the ordering procedure. Two illustrative examples and some concluding remarks are found in Sections 4 and 5, respectively.

2. On the notion of multivariate ordering

As is well known, a clear and unambiguous ordering of observations can be achieved only in a one-dimensional space. Indeed, if $x_1, x_2, \ldots, x_n$

are n observations in E^1, we can place them in increasing order as $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. For multivariate data, an ordering is usually obtained by moving from the given E^p to a suitable E^1. When all the multivariate points lie in a one dimensional subspace, i.e. on a straight line, the E^1 sub space is uniquely determined, and an unambiguous ordering is given. In all the other cases, the subspace E^1 is not unique and some information will be lost in the mapping from E^p to E^1.

Following the seminal work of Barnett (1976), several approaches for choosing the E^1 subspace can be adopted and different subordering principles arise.

Given a data matrix $\mathbf{X} = \{x_{ij}\}$, let X_j, $j = 1, \ldots, p$, denote the j-th variable, and $\mathbf{x}'_i = (x_{i1}, \ldots, x_{ip})'$ denote the set of the p measurements for the i-th statistical units, $i = 1, \ldots, n$. A simple, yet not very useful, ordering approach is the so-called *marginal ordering*, where the E^1 space is chosen to coincide with one of the p subspaces determined by one of the p variables X_j. The observations will be marginally ordered on the j-th variable as $x_{(1)j} \leq x_{(2)j} \ldots \leq x_{(n)j}$, and, hence, p different rankings will be obtained.

Nowadays, a very popular ordering approach is based on the notion of data depth. This is a type of *partial ordering*: the sample is partitioned into subgroups and the E^1 space is the associated space of depth function values of each subgroup. Such partial ordering induces a *center-outward ranking* of the observations (see from the first work of Tukey, 1975 to more recent works by Liu *et al.*, 1999 and Zhang, 2002, and references therein).

For the data we are dealing with (i.e. a set of units described by performance indicators) and for the main goal of benchmarking, the data depth approach is not adequate. Indeed, in our case the goal is not to order observations with respect to a center (the deepest point), but to achieve overall ordering of cases from the worst performance to the better ones.

More useful for our goals is the so-called *reduced ordering* (or *R-ordering*), which can be distinguished, according to Mardia (1976), into *distance* ordering and *projection* ordering. Distance ordering is however again not suited for our type of data. In fact, it implies the use of some generalized distance measure from a single fixed point (usually the center of the distribution) according to a specific metric, and the chosen E^1 space coincides with the one spanned by such a distance. This type of reduced ordering again yields a center-outward ranking and, as noted in Barnett (1976), it fails to reduce to the conventional ordering principle in one dimension. Another type of distance ordering is that based on the minimum spanning tree (*MST*) (Friedman and Rafsky, 1979) which yields two types of one-dimensional ranking: linear and radial. In both cases the E^1 space is given by the distances on the *MST*. For linear ranking the computational effort can be prohibitive for large data sets (Gentle, 2002). Moreover, the point chosen as the starting node cannot be always interpreted in terms of good or bad performance. Radial ranking produces a center-outward ranking, and hence has the drawbacks previously listed for our goals. Another type of reduced ordering, which resembles distance ordering, is based on convex hull volume variations (the E^1 space is the space of convex hull volumes): the convex hull of a data set iteratively identifies points lying on the boundary, the most extreme of which is the one whose removal from the data maximizes the reduction in the convex hull volume (Tukey, 1975; Eddy, 1985; D'Esposito and Ragozini, 1999). Such a procedure provides ordering of the data from the outside in, and in more than 3 dimensions could be computationally expensive.

In projection ordering each multivariate observation $\mathbf{x}'_i$ is reduced to a single value x_i^* by means of some combination of the x_{ij} component values or, equivalently, by an appropriate projection rule. The chosen E^1 space is the projection line. This approach does not have the disadvantage of distance ordering, and in the univariate case reproduces the natural ordering. Hence, it is well suited for our purposes, i.e. building an ordering criterion that ranks observations from "worst" performers to "best" ones.

The main issue here is to determine the direction $\mathbf{u}$ for such a projection. Commonly the first principal component is used. However, this strategy, too, does not completely meet the declared goals as it provides the most interesting direction in terms of variance and correlation without purposively considering any ordering criterion. Furthermore, it is not guaranteed that all performance indicators will be positively correlated with the first principal component. In particular, if subgroups of indicators are present, each one measuring different performance aspects, it is likely that the first principal component is correlated with the leading indicator subgroup. To avoid the first drawback, Korhonen and Siljamäki (1998) proposed to compute the ordinal principal component defined as a new ordinal variable which maximizes the sum of the squared rank-correlation with respect to the original variables. However, the procedure is very time consuming for real size problems, and also in this case the first ordinal component could be determined by only some leading indicators, neglecting the others.

Finally, with regard to projection ordering, it is worth noting that different criteria could yield very different rankings of the units. However, if the p indicators are strongly correlated to each other, different types of projections yield similar orderings.

3. The ordering procedure

Focusing our attention on projection ordering, we propose *i)* to determine a meaningful direction for ordering from the worst towards better cases, the *"worst-best" direction*, *ii)* then to project data on it, and *iii)* finally to rank our observations in the associated univariate space.

3.1. The "worst-best" direction

With the aim of determining the direction for the projection taking into account the benchmarking goals, we have first of all to define the cases which correspond to the "worst" and "best" performances, in the following $\mathbf{x}'_{worst}$ and $\mathbf{x}'_{best}$ respectively. These two cases represent kinds of ideal points combining all the "worst" or "best" achievable performances and indicators are at the lowest/highest possible values. Hence, in our framework the "worst" and "best" terms do not necessarily mean worst and best quality.

In order to construct them, we propose a data-driven strategy that relies on the use of convex hull and *archetypal points* (Cutler and Breiman, 1994). The latter represent a sort of pure individual types and are few points lying in the boundary of data scatter which synthesize the whole set of observed data. They are defined as m (with m chosen by the analyst) p-dimensional vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ whose convex combination $\sum_{k=1}^{m} \alpha_k \mathbf{a}_k$ best approximates the $\mathbf{x}_i'$'s by minimizing

$$\sum_{i=1}^{n} \left\| \mathbf{x}_i' - \sum_{k=1}^{m} \alpha_{ik} \mathbf{a}_k \right\|^2 \tag{1}$$

with $\alpha_{ik} \geq 0, \sum_{k=1}^{m} \alpha_{ik} = 1$.

The archetypes $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are shown to a be mixture of the $\mathbf{x}_i'$ data values, i.e.

$$\mathbf{a}_k = \sum_{i=1}^{n} \beta_{ki} \mathbf{x}_i', \qquad k = 1, \ldots, m \tag{2}$$

with $\beta_{ki} \geq 0, \sum_{i=1}^{n} \beta_{ki} = 1$.

For $m > 1$, the archetypes fall on the convex hull of the data. Given the properties of points lying on the convex hull, "they are extreme data values such that all the data can be well represented as a convex mixture of archetypes" (Cutler and Breiman, 1994).

When $m = 2$, the two most extreme two points on the convex hull of the data are selected, i.e. the two points having the maximum distance between them and the minimum distance with respect to the others. From a theoretical point of view, these two archetypes coincide with the idea of pure "worst" and "best" cases. Note that they are not necessarily actually observed, and lie on the border of the data region. In order to be sure that these two points correspond to the worst and best performers, the ranks on each variable of the two archetypes with respect to the other data values should be inspected. The "worst" archetype should have low values for most of the ranks, the opposite for the "best".

Another possibility to construct the "worst" and the "best" performers could rely on the notion of *multivariate extremes*, i.e. points that are the combinations of all the minima (or maxima) in the marginal orderings (Barnett, 1976). Hence, the vector of the minimum of each variable defines the "worst" performer

$$\mathbf{x}'_{worst} = \left(x_{(1)1}, ..., x_{(1)p}\right) \tag{3}$$

and, analogously, the vector of the maximum determines the "best" performer

$$\mathbf{x}'_{best} = \left(x_{(n)1}, ..., x_{(n)p}\right). \tag{4}$$

It is worth noting that for some performance indicators (e.g. unemployment rate, drop out rate, alumni/faculty rate,...) the minimum $x_{(1)j}$ does not correspond to the worst performance (and viceversa for the maximum). For such variables the ordering relation has to be reversed simply by multiplying the variable by (-1). It is easy to show that the two extreme points in (3) and (4) coincide with two vertices of the rectangular hull of the data, i.e. the minimum hyperparallelotope containing the data.

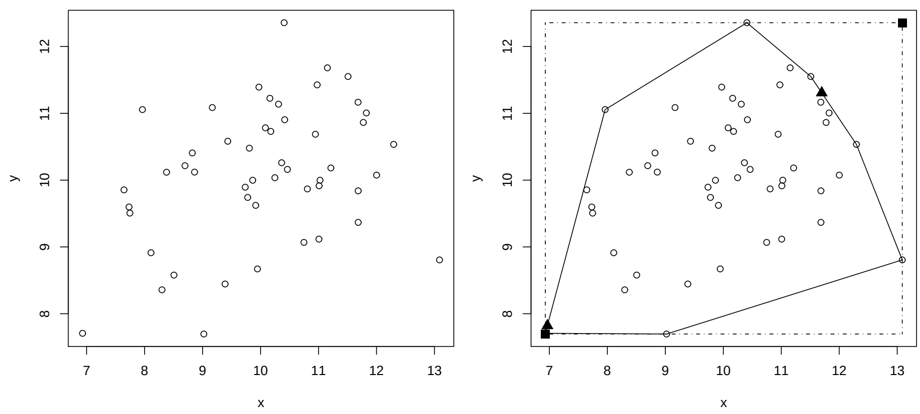
However, when outliers are present, the rectangular hull may change substantially. Hence, the "worst" and "best" performers, located on its vertices, could be too extreme as combinations and not admissible as real cases. To avoid this drawback, robust extreme values can be obtained either by taking the α and the $(1-\alpha)$ quantiles on each marginal indicator, i.e.

$$\begin{aligned}\mathbf{x}'_{worst} &= \left(x_{(\lfloor \alpha n \rfloor)1}, ..., x_{(\lfloor \alpha n \rfloor)p}\right) \\ \mathbf{x}'_{best} &= \left(x_{(\lceil (1-\alpha) n \rceil)1}, ..., x_{(\lceil (1-\alpha) n \rceil)p}\right),\end{aligned} \tag{5}$$

or by cleaning the data set by applying some peeling procedure (see for example the rectangular trimming for bivariate data (Nath, 1971; Dyer, 1973) or convex hull peeling procedures (Eddy, 1982; Porzio and Ragozini, 2000).

Even if the "worst" and "best" cases, constructed in this way, are very simple to interpret with a clear meaning even for nonstatisticians, and are very simple to compute, for particular data cloud shapes the rectangular hull could be completely misleading (Brooks, Carrol and Verdini,1988). On the contrary, with respect to $\mathbf{x}'_{worst} = \left(x_{(1)1}, ..., x_{(1)p}\right)$ and $\mathbf{x}'_{best} = \left(x_{(n)1}, ..., x_{(n)p}\right)$ in eqs. (3) and (4), the archetypal points are closer to the observed data and depend on the actual data cloud shape. Hence, we believe that the archetypes represent a better solution to determine the "worst" and "best" extremes.

In Figure 1(a), a set of 50 simulated bivariate normal data is shown along with the two pairs of "worst" and "best" performers chosen as the multivariate extremes $\mathbf{x}'_{worst} = \left(x_{(1)1}, ..., x_{(1)p}\right)$ and $\mathbf{x}'_{best} = \left(x_{(n)1}, ..., x_{(n)p}\right)$, or as the two archetypes $\mathbf{x}'_{worst} = \mathbf{a}_1$ and $\mathbf{x}'_{best} = \mathbf{a}_2$ (Figure 1(b)) .



(a) (b)

Figure 1. A set of 50 simulated bivariate normal data (a). The rectangular hull (dashed line) and the convex hull (solid line) are superimposed, along with two pairs of "worst" and "best" performers chosen as multivariate extremes in eqs. (3) and (4) ($squares$), and as the archetypes in eq. (1) ($triangles$) (b).

3.2. The projection and ordering

Given the two extreme values $\mathbf{x}'_{worst}$ and $\mathbf{x}'_{best}$, the next step is to determine a direction $\mathbf{u}'$, with $\mathbf{u} \in E^p$ and $\mathbf{u}'\mathbf{u} = 1$, on which to project the observations $\mathbf{x}'_i$. The set of projected values x_i^* will be the associated one-dimensional subspace for ordering.

Given the vector $\mathbf{x}'_{w\to b}$ joining the two extreme performers, $\mathbf{x}'_{w\to b} = (\mathbf{x}'_{worst} - \mathbf{x}'_{best})$, the projection of $\mathbf{x}'_i$ on the $\mathbf{x}'_{w\to b}$ vector will be:

$$x_i^* = \frac{\mathbf{x'}_i \cdot \mathbf{x}_{w\to b}}{\|\mathbf{x}_{w\to b}\|} = \mathbf{x'}_i \cdot \frac{(\mathbf{x}_{worst} - \mathbf{x}_{best})}{\|\mathbf{x}_{worst} - \mathbf{x}_{best}\|} = \mathbf{x'}_i \cdot \mathbf{u} \tag{6}$$

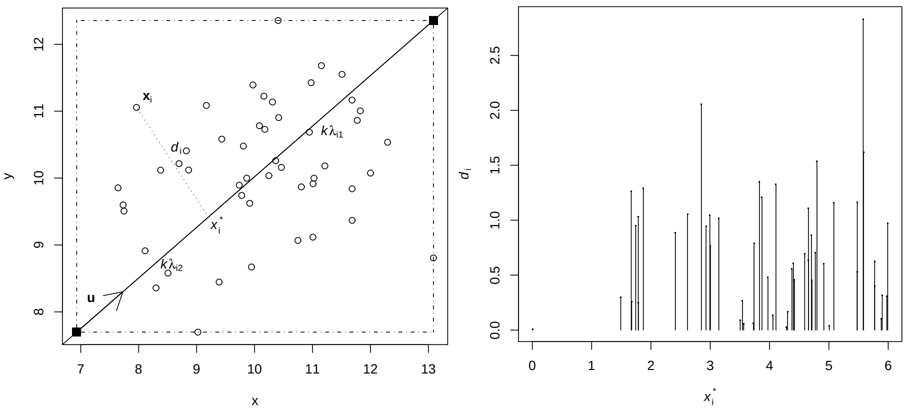
where the vector $\mathbf{u}' = \frac{(\mathbf{x'}_{best} - \mathbf{x'}_{worst})}{\|\mathbf{x'}_{best} - \mathbf{x'}_{worst}\|}$ is the direction cosines vector of $\mathbf{x'}_{w\to b}$, and where $\|\cdot\|$ is the euclidean norm.

Ordering along the "worst-best" direction will be obtained by simply considering the ranks r_i of the x_i^*'s. Each projected point can be written as a convex combination of $x^*_{(1)}$ and $x^*_{(n)}$, respectively the minimum and maximum values of the projected data, i.e. $x_i^* = \lambda_{1i} x^*_{(1)} + \lambda_{2i} x^*_{(n)}$ with $\lambda_{1i} + \lambda_{2i} = 1$, $i = 1, \ldots, n$. It is worth noting that $x^*_{(1)}$ and $x^*_{(n)}$ do not necessarily coincide with the projections of $\mathbf{x'}_{worst}$ and $\mathbf{x'}_{best}$. The coefficients λ_{i1} and λ_{i2} express the weights of the two extreme cases in the composition of the x_i^* values (Figure 2(a)). If the $\mathbf{x'}_{worst}$ and $\mathbf{x'}_{best}$ are the two archetypes, the coefficients λ_{1i} and λ_{2i} coincide with the α_{1i} and α_{2i} coefficients in 1.

When reduced ordering based on projection is used, much information about the original multivariate structure, such as data cloud shape or existence of outlying data, could be lost by simply looking at the ranks r_i of the ordered data $x^*_{(i)}$. Part of this information can be recovered by looking at the plot of the d_i versus x_i^*, where d_i is the Euclidean distance of a point $\mathbf{x}_i$ from the straight-line passing through the vector $\mathbf{x'}_{w\to b}$ (Figure 2(b)).

It has to be noted that, in the case of the archetypal definitions for $\mathbf{x}'_{worst}$ and $\mathbf{x}'_{best}$, the direction $\mathbf{u}'$ coincides with a diagonal of the convex hull of the data. Such an interpretation of $\mathbf{u}'$ suggests an advantage of our reduced ordering procedure over the one which uses the first principal component. Indeed, for spherical or non-elliptical data clouds the worst-best ordering direction can always be determined, whilst this is not true for the principal components.

Finally, if data are transformed, as is customary in many applications, attention should be paid to the effect on ordering. Convex hull (and the rectangular hull too) are affine invariant, hence none of the linear transformations affects the "worst" and "best" cases construction. On the contrary nonlinear transformations modify the data cloud shape, and convex hull will also change.



(a) (b)

Figure 2. A set of 50 simulated bivariate normal data. The rectangular hull (dashed line) along with two pairs of "worst" and "best" performers chosen as in (3) and in (4) ($squares$), the projection direction **u**, *a projected point x_i^*, the distance d_i, λ_{i1} and λ_{i2} (a). The distances d_i versus the ordered projected points $x_{(i)}^*$ are plotted (b).*

4. Two illustrative examples

In order to show how our ordering procedure works, we apply it to two real data sets. The first consists of six demographic indicators related to age and family structure for the 103 Italian provinces (Figure 3). The indicators were constructed through elaboration of data from the 2001 Italian census, and namely are: average family size (*AFS*), incidence of families with 5 and more members (*Fam5+*), incidence of couples with children (*CwK*), ratio between the number of couples legally married and the number of couples not married (*MC/nMC*), ratio of old people to kids (*H/K*), and, finally, the oldness indicator(*HI*). The indicators have been scaled with respect to the interquartile range in order to take into account the different measurement unit and variability.

We know that nowadays the Italian population is tending to decrease in size and is aging, with a strong tendency towards singleness or to "empty cradles". Such a situation is producing alarming forecasts for Italy's population decline. In such a case, without assigning any positive or negative meaning to the words “worst” and “best”, we consider as “best” the provinces that are younger and with larger families as they profile provinces with increasing populations. Hence, in the following, we revert the order of the last two variables by considering $-H/K$ and $-HI$. From the scatterplot matrix (Figure 3) the data cloud appears to be very elongated in most of the directions, with some nonlinear structures in the others. Instead, some variables are highly correlated.

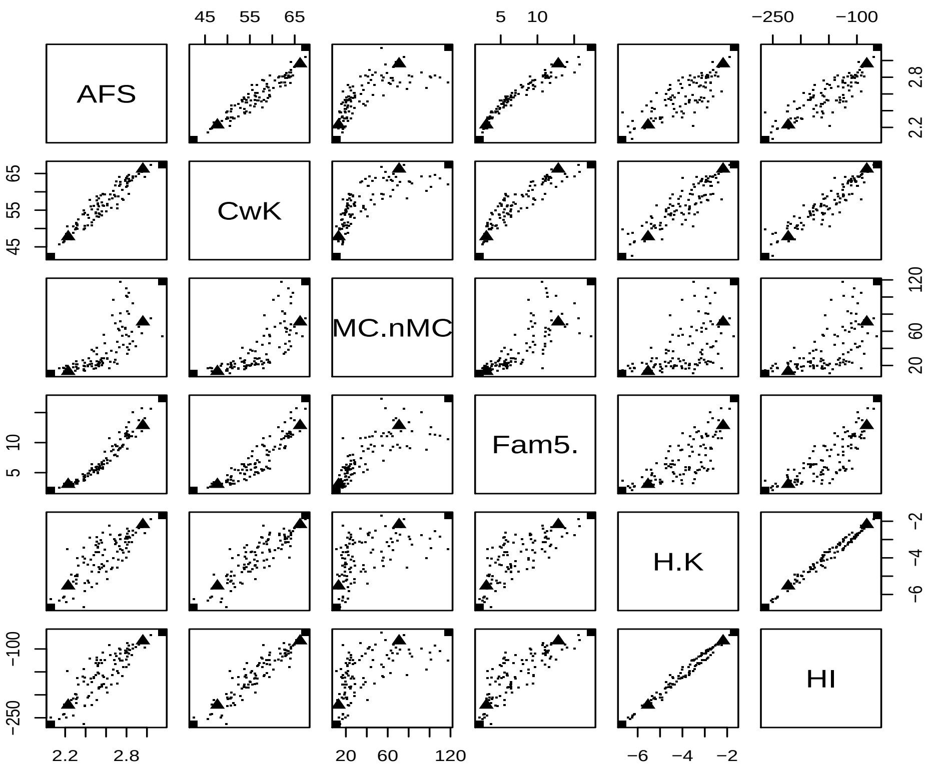
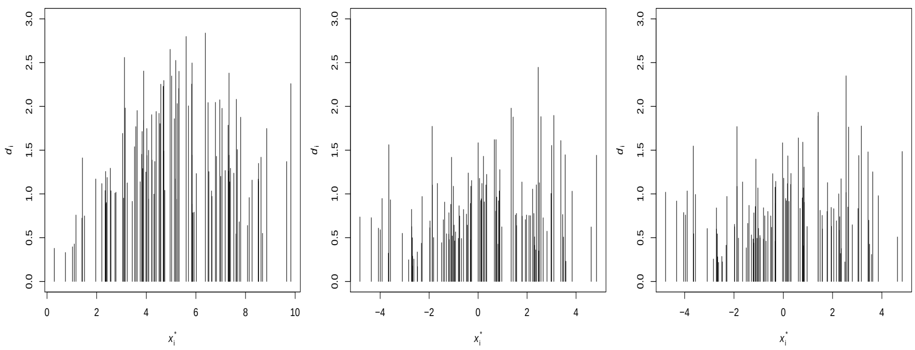


Figure 3. Six demographic indicators from 2001 the Italian Census.

We adopt the two data-driven constructions for the worst and best performances (in Figure 3 the multivariate extremes are represented as squares, and the archetypes as triangles).

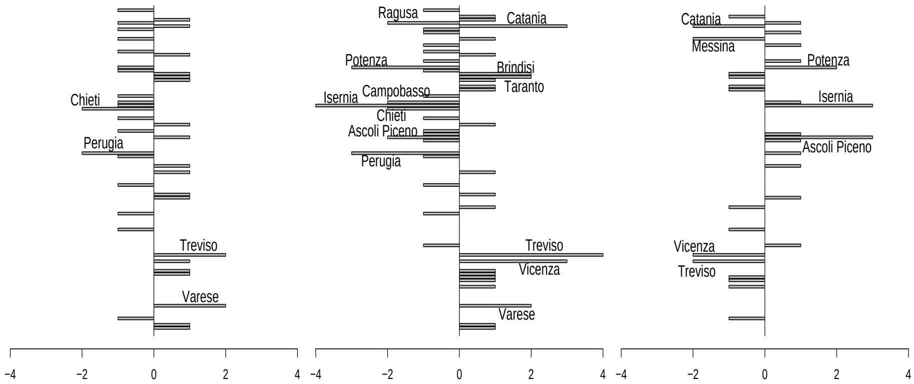
The squares lie in the corners of the data scatter and the “worst” case combines the *AFS*, the *CwK* and the *Fam5+* values of Trieste, the value of *MC/nMC* of Aosta, the values of $-H/K$ and $-HI$ of Ferrara. On the contrary the "best" case combines the *AFS*, the $-H/K$, the $-HI$*CwK* and the *Fam5+* values of Naples, the value of *CwK* of Caserta and the *MC/nMC* value of Potenza. Figure 3 exhibits that when a linear data structure is present the archetypes and the multivariate extremes are on the same direction, but the archetypes are in an inner position closer to the majority of the data. On the other hand, in the case of a non linear structure archetypes and extremes lie on different directions, and archetypes continue to be closer to the data. This feature is highlighted also by the d_i's distance behaviour plotted in Figure 4 for each of the three ordering procedures.



(a) (b) (c)

Figure 4. Distance plots for three orderings obtained starting from multivariate extremes (a), archetypes (b) and first principal component (c).

Denoting with R_{extr}, R_{arc} and R_{ACP} respectively the ranking obtained starting from the multivariate extremes, the archetypes and first principal component, we compare the three rankings considering the pairwise differences (Figure 5) and the Kendall rank-correlation coefficients. Looking at the differences it may be noted that three rankings mostly agree with each other. For few provinces (highlighted in the graphics) the differences in the ranks are greater than 2. The R_{arc} and R_{ACP} seem to be more similar with respect to the R_{extr}. The three rank-correlation coefficients are all 0.999. Note that the provinces with the higher differences are more or less the same in the three plots. These results agree with what was expected given the data cloud structure in Figure 3 which is mostly linear in many of the p dimensions of the Euclidean space.



(a) $R_{arc} - R_{ACP}$ (b) $R_{arc} - R_{extr}$ (c) $R_{extr} - R_{ACP}$

Figure 5. Comparison of three orderings for archetypes (R_{arc}), multivariate extremes (R_{extr}) and first principal component (R_{ACP}).

The second data set consists of the average bank deposit (BD) and the unemployment rate (UR) (reverted in the order) for the 103 provincial capitals in Italy (Figure 6). Also in this case the data have been scaled by the interquartile range. The data appear to have a highly nonlinear structure. In Figure 6 the two data driven solutions for the "best" and "worst" cases are superimposed. In the case of the multivariate extremes, the "worst" case corresponds to a hypothetical town with $UR = 30.53$ (Reggio Calabria) and $BD = 3418.24$ (Vibo Valentia), while the "best" will correspond to a town with $UR = 1.71$ (Lecco) and $BD = 20890.66$ (Milan). It clearly appears that the bivariate extreme located in the upper right corner is very far away from the majority of the data, and heavily influences the ordering direction, whilst the two archetypes lie close to the body of the data. Due to the data cloud shape, even if there are only two dimensions, overall ordering is difficult to sight and marginal ordering could be misleading. By applying the proposed procedure to the values we rank the 103 towns.

Figure 7, which portrays the d_i values for the three ordering procedures, highlights the nonlinear structure of data, the extremeness of Mi-

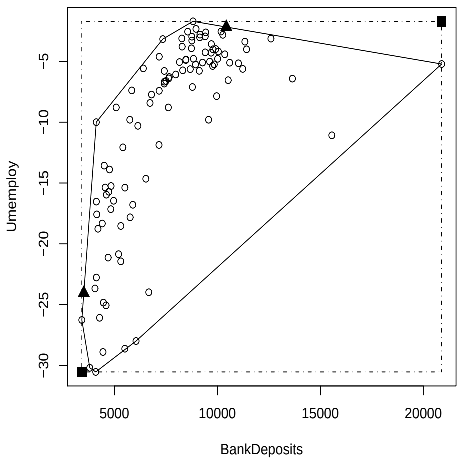
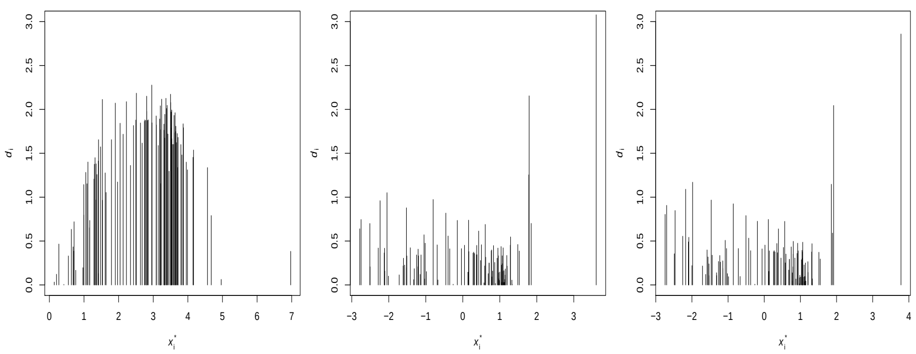


Figure 6. 103 Italian provinces by average bank deposit and unemployment rate. The rectangular hull (dashed line) and the convex hull (solid line) are superimposed, along with two pairs of "worst" and "best" performers chosen as in eqs. (3) and (4) ($squares$), and as the archetypes in eq. (1) ($triangles$).

lan and some clusters of towns. However, looking at the d_i's in Figure 7(a), it is worth noting that the ranking based on the multivariate extremes is influenced by some outliers yielding overall high distances, while the archetypes and the first principal component present overall low distances except the ones corresponding to the far away data.

Looking at the differences between rankings (Figure 8), we note that they are greater than in the first example. As noted in the plot of d_i's, R_{arc} and R_{ACP} are more similar with respect to R_{extr}: only 7% of differences are equal to or greater than 2 in absolute value. On the other hand, on comparing the R_{extr} with R_{arc} and R_{ACP}, respectively 53% and 45% of differences are equal or greater than 2 in absolute value. The provinces with the higher differences are more or less the same in the three comparisons. If the rank-correlation is used for comparison, the values ($\rho_{arc,ACP} = 0.999$, $\rho_{arc,extr} = 0.996$, $\rho_{extr,ACP} = 0.994$) appear to be very similar in contrast with what is evident from the graphics of individual differences. This could suggest that procedures based on rank-correlation, like the principal ordinal component, could not capture non



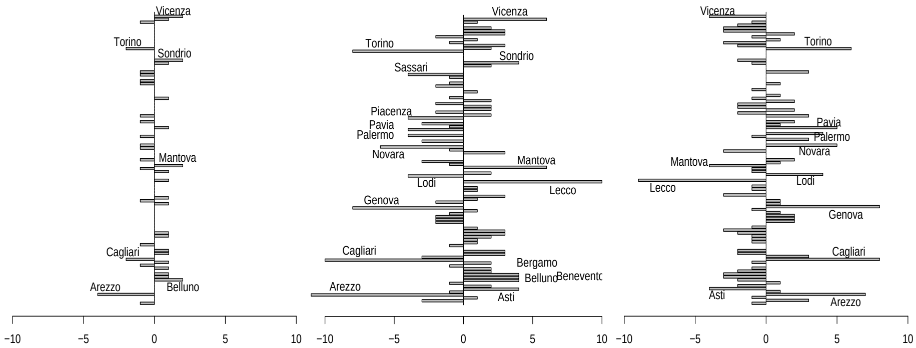
(a) (b) (c)

Figure 7. Distance plots for three orderings obtained starting from multivariate extremes (a), archetypes (b) and principal component(c).

regular data structures.

5. Prospects

The proposed ordering method based on the archetypes seems to meet the aim of obtaining an ordering that is simple to interpret and computationally not expensive. Even if in the two proposed examples the archetype-based ranking is similar to that based on the first principal component, the archetype ordering direction has a simple meaning in terms of ranking, and avoids the possible drawbacks of principal components. On the other hand, archetype-based ranking differs from the ones based on multivariate extremes in the case of complex data structures, and it seems to be close in case of linear data structures. Further comparisons with other ordering criteria will be performed in future works through some simulation studies. At the same time, the relationships with the composite indicator approach to ordering will be explored. As a matter of fact, the $R - ordering$ reduces all the performance indicators into a linear combination with weights given by the components of the $\mathbf{u}$. Finally, the possibility to use nonlinear projection criteria and a modified version of the archetypes are under study.



(a) $R_{arc} - R_{ACP}$ (b) $R_{arc} - R_{extr}$ (c) $R_{extr} - R_{ACP}$

Figure 8. 103 Italian provinces by average bank deposit and unemployment rate. Comparison of three orderings for archetypes (R_{arc}), multivariate extremes (R_{extr}) and principal component (R_{ACP}).

Acknowledgements: The authors would like to thank Dr. Domenico Vistocco for the MATLAB code to compute archetypes.

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