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Conditional least squares and model-based bootstrap inference in bilinear models

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Summary: In this paper the use of two model-based bootstrap procedures, parametric bootstrap and residual bootstrap, is implemented to get an estimate of the sampling distribution of the conditional least squares estimators. Some results of a Monte Carlo experiment, that give insight about the performance, in small samples, of the proposed approaches are also discussed.

Key words: Bilinear models, Bootstrap, Conditional least squares, Monte Carlo.

1. Introduction

Non linear parametric modelling of time series is a promising approach both in theoretical and applied time series analysis (Priestley, 1988; Tong, 1990 *inter alia*). It is well known that Gaussian linear time series models fail to capture characteristics commonly observed in practice such as asymmetry between the ascent and descent periods of the series. Non linear modelling moves time series analysis a step closer to reality improving forecasting accuracy.

Bilinear models are the easiest way to introduce a non linear structure in time series analysis modelling (Subba Rao, 1981). This class is obtained by adding to an ARMA structure interaction components between the observed series and the innovations. In this way a class of models is defined which is able to describe stationary level and occasional sharp spikes, typical of environmental and financial time series. For this class of models, probabilistic properties concerning stationarity, invertibility and ergodicity have been proved (see Liu and Brockwell, 1988 *inter alia*). However, identification and inferential problems seem to be more difficult to cope with. For stationary and ergodic time series several different approaches are already avaliable in the literature for the estimation of bilinear models. Unfortunately, their nonlinear structure makes quite complex to study the sampling properties of these estimators analytically. For sub-classes of bilinear models some asymptotic results, such as consistency, are available but often very little is known about the sampling distribution of the estimators involved. For the conditional least squares estimator considered in this paper, it is known that under mild conditions on the higher moments of the process the sampling distribution converges to a Gaussian distribution. However, no closed-form expression is known for the variance-covariance matrix of the estimators and, therefore, no use of this result can be done for inferential purposes.

In this paper we propose to use model based bootstrap procedures to get an estimate of the sampling distribution of the conditional least squares estimators. The paper has the following structure. In section 2 the estimation of bilinear models is briefly reviewed. In section 3 the use of two model based bootstrap inference procedure, parametric bootstrap and residual bootstrap, is discussed to approximate the sampling distribution of the conditional least squares estimators in a wide class of bilinear models. In section 4 some results of a Monte Carlo simulation, performed to study the performance of the proposed procedure in small samples, are reported. Some concluding remarks are provided in section 5.

2. Conditional least squares estimation of bilinear models

The general form of a bilinear time series $\{X_t, t = 0, \pm 1, \pm 2, \ldots\}$ denoted by $BL(p, q, k, r)$, is defined by

$$X_t = a_0 + \sum_{i=1}^{p} a_i X_{t-i} + \sum_{j=1}^{q} c_j \varepsilon_{t-j} + \sum_{i=1}^{k} \sum_{j=1}^{r} b_{ij} \varepsilon_{t-i} X_{t-j} + \varepsilon_t$$

where $\{\varepsilon_t\}$ is an *iid* sequence of random variables (innovations) with zero mean and common variance σ^2. For stationary and ergodic bilinear time series models, estimation has received considerable attention in the literature but, unfortunately, asymptotic properties are known only in some restrictive models or cannot be used for inferential purposes. The most popular methods for estimating the parameters of the bilinear models are the least squares method (Guegan and Pham, 1989) and the method of moments (Kim *et al.*, 1990). But even for the first order bilinear model $BL(0,0,1,1)$, the simplest specification, nothing is yet known about the limiting distributions of the least squares estimators. If attention is restricted to Gaussian innovations, estimators obtained by the method of moments are strongly consistent and asymptotically normally distributed only if $q = 0$. For the Gaussian innovation case maximum likelihood estimators have been also derived (see Gabr, 1993) but only for the special bilinear model $BL(p,0,p,1)$.

The most interesting and promising estimation approach seems to be the conditional least squares approach (COLS) proposed by Grahn (1995). The COLS estimators have nice asymptotic properties and can be computed in a quite general class of bilinear models specified as

$$X_t = \sum_{i=1}^{p} a_i X_{t-i} + \sum_{j=1}^{q} c_j \varepsilon_{t-j} + \sum_{i=1}^{k} \sum_{j=w}^{r} b_{ij} \varepsilon_{t-i} X_{t-j} + \varepsilon_t \qquad (1)$$

where $w = \max\{q,k\} + 1$. The process $\{X_t\}$ is assumed to be stationary, ergodic, causal and square integrable solution of the equation (1). The process $\{\varepsilon_t\}$ is assumed to be such that $E(\varepsilon_t^3) = 0$ and $E(\varepsilon_t^4) < \infty$.

The COLS estimation procedure for the parameter vector $\left\{a_1,\ldots,a_p,c_1,\ldots,c_q,b_{1w},\ldots,b_{kr},\sigma^2\right\}$ can be sketched as follows (see Grahn, 1995 for details).

Step 1. The coefficients of the AR component $a_1,a_2,\ldots,a_p$ are estimated by the Yule-Walker equations.

Step 2. The conditional least squares method is used to estimate proper quantities, functions of $\left\{c_1,\ldots,c_q,b_{1w},\ldots,b_{kr},\sigma^2\right\}$, which are uniquely determined by the conditional second order moments of the AR residual process $v_t = X_t - \sum_{i=1}^{p} a_i X_{t-i}$.

Step 3. Finally, from the estimates of the previous step, the estimates for the parameters $\left\{c_1,\ldots,c_q,b_{1w},\ldots,b_{kr},\sigma^2\right\}$ are identified.

Under mild assumptions on the existence of higher moments of the bilinear process, one can prove that the COLS estimators are strongly consistent, asymptotically normally distributed and the law of the iterated logarithm applies.

Unfortunately, even if in principle it should be possible to compute the asymptotic variance-covariance matrix, no closed-form expression for this matrix has been so far provided in the literature (Grahn, 1995, p. 517). This is quite usual in nonlinear time series analysis where, due to the complexity of the models involved, approaches based on analytic derivations become very soon extremely difficult. In this context, resampling techniques, such as the bootstrap, can be effectively used to consistently estimate both the sampling distribution of the estimators involved and its limiting normal distribution. Moreover, it is well known that the use of the bootstrap allows greater accuracy than estimated normal approximations.

3. Model based bootstrap inference in bilinear models

In their classical form, as first proposed by Efron (1979), bootstrap methods are designed for application to samples of independent data. Under that assumption, they implicitly produce an adaptive model for the marginal sampling distribution. Extensions to dependent data are not straightforward and modifications of the original procedures are needed in order to preserve the dependence structure of the original data in the bootstrap samples. In the context of stationary time series models two alternative groups of techniques are available.

On one hand, nonparametric, model-free bootstrap schemes have been proposed. In these procedures blocks of consecutive observations are resampled randomly with replacement, from the original time series and assembled by joining the blocks together randomly in order to obtain a simulated version of the original process (Kunsch, 1989; Politis and Romano, 1992 *inter alia*). These approaches, known as blockwise bootstrap or moving-block bootstrap, generally works satisfactorly and enjoys the properties of being robust against misspecified models. However, the resampled series exhibits spurious features which are caused by randomly joining selected blocks. The major drawback with these block schemes is that the dependence between different blocks is neglected in the resampled series. As a consequence, the asymptotic variance-covariance matrices of the estimators based on the original series and those based on the bootstrap series are different and a modification of the original scheme is needed. A possible solution is the matched moving-block bootstrap, (Carlstein *et al.*, 1996), based on a quite complex procedure which resamples the blocks according to a Markov chain whose transitions depend on the data. A further difficulty, is that the bootstrap sample is not (conditionally) stationary. This can be overcome by taking blocks of random length, as proposed by Politis and Romano (1994), but a tuning parameter, which seems difficult to control, has to be fixed. Morever, a recent study of Lahiri (1999) shows that this approach is much less efficient than the original one.

On the other hand model-based approaches have been proposed. In this case, the dependence structure of the series is modelled explicitly and the bootstrap sample is drawn from the fitted model. Freedman (1984) and Bose (1988) introduced this scheme for autoregressive models and Kreiss and Franke (1992) for ARMA models. A model based procedure runs as follows: (*i*) fit a suitable model to the data; (*ii*) construct residuals from the fitted model; (*iii*) generate new pseudo-series by incorporating random samples of the residuals in the fitted model. Obviously, these procedures are sensitive to model misspecification, in which case they lead to bootstrap estimators which are not consistent. This is the major drawback of these schemes especially when both the parameters of a model and its structure have to be identified from the data. This problem is less serious when the model is selected by strong subject-matter considerations or when the selection is well supported by extensive data. In this case model based bootstrap schemes are more efficient than nonparametric ones (see Horowitz, 1995) and so they could be a straightforward choice.

A model based scheme for bilinear time series can be constructed as follows. Assume that a bilinear model as specified in equation (1) is given. Let $x_1,\ldots,x_n$ be the observed time series and let $x_{1-p},\ldots,x_0$ be the initial conditions.

1. By COLS method obtain an estimate of the parameter vector $\hat{\theta}=\left(\hat{a}_1,\ldots,\hat{a}_p,\hat{c}_1,\ldots,\hat{c}_q,\hat{b}_{1w},\ldots,\hat{b}_{kr},\hat{\sigma}^2\right)$.
2. Obtain an estimate $\hat{F}_\varepsilon$ for F_ε, the distribution of the residuals. Generate $\varepsilon^*_{-q},\ldots,\varepsilon^*_0,\varepsilon^*_1,\ldots,\varepsilon^*_n$ independently from $\hat{F}_\varepsilon$.
3. Given $\left\{x^*_{1-2p},\ldots,x^*_{-p}\right\}=\left\{x_{1-p},\ldots,x_0\right\}$ generate a boostrap pseudo-series as

$$X^*_t=\sum_{i=1}^{p}\hat{a}_iX^*_{t-i}+\sum_{j=1}^{q}\hat{c}_j\varepsilon^*_{t-j}+\sum_{i=1}^{k}\sum_{j=w}^{r}\hat{b}_{ij}\varepsilon^*_{t-i}X^*_{t-j}+\varepsilon^*_t$$

   with $t=1,2,\ldots,n$.
4. By COLS obtain the bootstrap counterpart $\hat{\theta}^*$ of $\hat{\theta}$.
5. Approximate the sampling distribution of $\hat{\theta}_i$ with the bootstrap distribution of $\hat{\theta}^*_i$.

As usual, the bootstrap distribution can be approximated numerically (by Monte Carlo simulation) repeating B times steps 2–4. The empirical distribution function of the bootstrap replicates

$\hat{\theta}^*_{i1}, \ldots, \hat{\theta}^*_{iB}$ can be used to study the sampling properties of the estimator $\hat{\theta}_i$ or to construct confidence intervals for θ_i.

Estimation of the residual distribution, in step 2, can be obtained in different ways. If it is reasonable to assume for it some kind of parametric model, that is $F_\varepsilon \equiv F_{\varepsilon,\omega}$, a parametric bootstrap scheme can be used. In this case bootstrap innovations can be generated from $F_{\varepsilon,\hat{\omega}}$, with $\hat{\omega}$ estimated from the data at hand. A typical assumption is that all the nonlinear structure present in the data is correctly captured by the specified model, hence the innovations can be assumed Gaussian, that is $F_{\varepsilon,\omega} \equiv N(0, \sigma^2)$. Thus, bootstrap innovations can be generated from $N(0, \hat{\sigma}^2)$.

If no parametric model for the residuals is plausible residual bootstrap can be employed. In this case the distribution of the residuals is estimated by $\hat{F}_\varepsilon(x) = n^{-1} \sum_{t=1}^{n} I(r_t - \bar{r} \le x)$ where $\bar{r} = n^{-1} \sum_{t=1}^{n} r_t$ and $I(\cdot)$ is the indicator function and bootstrap innovations can be generated from $\hat{F}_\varepsilon$.

The good asymptotic properties of COLS estimators (strong consistency, asymptotic normality and law of iterated logarithm) and the structure of the parameters involved (which can be viewed as functions of means) ensure the minimal regularity conditions for the validity of the bootstrap schemes previously discussed (Kunsch, 1989).

Confidence intervals based on the bootstrap distribution, can be constructed in several different ways. The simplest way is to exploit the asymptotic Gaussian distribution of the estimators by using the bootstrap just to estimate this limiting distribution. In this case an approximated confidence interval of nominal level $1-2\alpha$ can be constructed as

$$\left[\hat{\theta}_i \pm z_\alpha \sqrt{Var^*(\hat{\theta}_i)}\right]$$

where z_α is the α-quantile of the standard Gaussian distribution,

$$Var^*\left(\hat{\theta}_i\right) = (B-1)^{-1} \sum_{j=1}^{B} \left(\hat{\theta}_{ij}^* - \bar{\hat{\theta}}_i^* \right)^2$$

is the bootstrap variance estimator and $\bar{\hat{\theta}}_i^* = B^{-1} \sum_{j=1}^{B} \hat{\theta}_{ij}^*$.

Alternatively, the bootstrap percentile method of Efron (1981) can be employed and an approximate equal-tailed confidence interval of nominal level $1-2\alpha$ is given by $\left[\hat{\theta}_i^{*(\alpha)}, \hat{\theta}_i^{*(1-\alpha)}\right]$ where $\hat{\theta}_i^{*(\alpha)}$ is the α-quantile of the bootstrap distribution obtained as the solution to $(B+1)^{-1} \sum_{j=1}^{B} I\left(\hat{\theta}_{ij}^* \leq \hat{\theta}_i^{*(\alpha)}\right) = \alpha$.

Unfortunately others approaches, which are known to be more accurate, are difficult to use. The bootstrap-*t* percentile requires a closed-form expression for the variance-covariance matrix of the estimators and so it is not feasible in this case. The BCa method requires the estimation of a tuning parameter which is usually obtained by the influence function which is not known in our case where a complex, multi-step, estimator is involved. Numerical approaches to improve accuracy such as Loh correction (Loh, 1987) or double bootstrap can be used in principle but they can be extremely time consuming.

4. Simulations results

To investigate the *performance* of the proposed procedures in small samples a simulation experiment has been performed. The simple model $X_t = b\varepsilon_{t-1}X_{t-2} + \varepsilon_t$, with $\{\varepsilon_t\}$ independent random variables distributed as $N\left(0, \sigma^2\right)$, was considered. The model structure is quite simple but it allows to identify easily all the necessary regularity conditions. If $b^2\sigma^2 < 1$ the process is stationary and if $b^8\sigma^8 < 1/105$ the COLS estimators $\left(\hat{b}, \hat{\sigma}^2\right)$ for the parameters $\left(b, \sigma^2\right)$ are asymptotically normally distributed. Moreover, this simple bilinear structure can be used to model residuals in more complex models such as regression models with bilinear residuals.

We fixed $\sigma^2 = 1$ and let b varying from 0.0 to 0.6. The value $b = 0.6$ violates the condition for the asymptotic convergence to the normal distribution (which is $b < 0.559$ in this case). Three sample sizes n = 200, 500 and 1,000 were used.

To get a deeper knowledge of the experimental context we estimated, by a Monte Carlo experiment of 50,000 runs, the sampling distribution of the estimators involved. Both calssical and resistant statistics were computed (Tab. 1 and Tab. 2). Due to the nature of the bilinear model, where interactions components are present, we found in the sampling distributions many outlying and even very extreme outlying values, especially when b increases approaching the non-stationary region. This is quite evident from the increasing difference between non robust and robust indexes and from the sharp increase in the range of the distributions. When the value of b increases we observe an increasing bias for the estimator of b and an increasing skewness for the sampling distribution. The skewness is even greater if we consider the non robust index, which makes evident the presence of a long right tail. When looking at the estimator of the variance we observe almost no bias and almost no skewness. Therefore, we expect that a bootstrap estimate of the whole distribution should work better than the estimated normal distribution for the approximation of the sampling distribution of $\hat{b}$. On the contrary, this estimated asymptotic normal distribution should work quite well for the sampling distribution of $\hat{\sigma}^2$.

In Fig. 1 we reported parallel box-plots of the distribution of the difference $\Delta v = Var^*(\hat{b}) - Var(\hat{b})$ where $Var(\hat{b})$ is the "true" variance, estimated by 50,000 Monte Carlo runs, and $Var^*(\hat{b})$ is its bootstrap estimate, for different values of b and for different time series length. In Fig. 2 we reported the distributions of $\Delta v = Var^*(\hat{\sigma}^2) - Var(\hat{\sigma}^2)$. Due to scale effects, we did not report extreme outlying values, those with a distance greater than 3 times the interquartile range from the first and the third quartile. In both cases, increasing b, we observe distribution with increasing bias and greater number of outlying observations. Estimation obtained by parametric bootstrap (on the left of each sub-panel) and the residual bootstrap are quite similar even if residual bootstrap gives more median-biased estimates. In any case both bootstrap estimators improve their performance as the time series length increase.

We also compared the results for the parametric bootstrap (PB) and the residual bootstrap (RB) approaches for confidence intervals based on both the normal approximation (NBCI) and on the Efron's percentile method (BPCI), we considered 1,000 simulation runs and 999 bootstrap replicates for each run. The nominal confidence level has been fixed to 0.90.

The empirical coverages for b are reported in Tab. 3. Confidence intervals based on the bootstrap percentile present a lower coverage error than the normal based ones. The bootstrap distribution succeeded in capturing the asymmetric behaviour of the sampling distribution. When comparing the two bootstrap schemes, as expected parametric bootstrap works better than residual bootstrap. However, residual bootstrap gives reasonable coverage errors especially for parameters not too close the limit of the region of asymptotic normality and for time series not too short. For both procedures, coverage error increases as b approaches the border of the region of convergence to the normal limit.

If we look at the median and the MAD of the length of the confidence intervals we see that again the BPCI gives better results. The intervals have a shorter length and are less variable. In any case the length and its dispersion increase as the parameter b gets close to the non- stationary region (see Tab. 5).

The empirical coverages for σ^2 are reported in Tab. 4. Here the results are less stable and with different behaviour with respect to the parameter b. When using the BPCI in the residual bootstrap scheme we find sensitive coverage errors which do not disappear increasing the sample size. The other schemes works quite well with low coverage errors. All the bootstrap schemes considered have a similar length distribution (see Tab. 5).

Similar results, not reported here, have been obtained for a nominal confidence level of 0.95.

5. Concluding remarks

In this paper we implemented two different model-based bootstrap schemes to estimate the sampling distributions of conditional least squares estimators for bilinear models. Because of the complexity of the estimators involved, no analytical inference procedure is available in this case. Bootstrap approaches, instead, seem to offer an effective and computationally efficient solution to this problem. Also, their relative performance tracks rather well our *a priori* expectations. Parametric bootstrap gives better results than residual bootstrap and Efron's bootstrap percentile confidence intervals are better, both for the empirical coverage and for the median length, than those based on the normal approximation with bootstrap variance estimation.

In almost all cases the performance improves greatly for increasing sample size.

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Appendix. Tables and figures.

Table 1. *Descriptive statistics for the sampling distribution of* $\hat{b}$ *based on 50,000 Monte Carlo runs. Experimental model:* $X_t = b\, X_{t-2}\, \varepsilon_{t-1} + \varepsilon_t,\ \varepsilon_t \sim N\left(0, \sigma^2\right)$ *with* $\sigma^2 = 1$.

b	*n*	*Mean*	*Med*	*SD*	*MAD*	*SK*	*RSK*	*Range*
0.0	200	0.000	0.000	0.071	0.086	-0.001	-0.006	0.655
	500	0.000	0.000	0.045	0.055	-0.025	-0.005	0.389
	1000	0.000	0.000	0.032	0.040	-0.002	-0.007	0.276
0.1	200	0.098	0.095	0.075	0.089	0.352	0.020	0.793
	500	0.099	0.098	0.048	0.058	0.215	0.014	0.479
	1000	0.100	0.099	0.034	0.042	0.155	0.023	0.294
0.2	200	0.195	0.187	0.088	0.101	0.775	0.055	1.070
	500	0.198	0.195	0.056	0.067	0.531	0.040	0.820
	1000	0.199	0.197	0.040	0.048	0.362	0.040	0.465
0.3	200	0.291	0.277	0.110	0.118	1.326	0.083	1.585
	500	0.297	0.289	0.072	0.081	1.016	0.067	1.268
	1000	0.298	0.294	0.051	0.059	0.692	0.050	0.770
0.4	200	0.382	0.360	0.140	0.140	1.972	0.108	2.531
	500	0.393	0.380	0.095	0.099	1.643	0.091	1.461
	1000	0.396	0.388	0.069	0.075	1.224	0.078	1.001
0.5	200	0.465	0.432	0.177	0.163	2.649	0.135	6.735
	500	0.484	0.462	0.132	0.122	5.188	0.117	7.321
	1000	0.491	0.476	0.097	0.096	2.155	0.110	2.018
0.6	200	0.527	0.484	0.220	0.184	4.989	0.151	15.881
	500	0.558	0.525	0.171	0.146	1.595	0.146	11.601
	1000	0.573	0.547	0.139	0.121	4.098	0.138	5.787

Legenda. *Med=Median, SD=Standard Deviation, MAD=Median Absolute Deviation, SK=Skewness, RSK=Resistant Skewness.*

Table 2. *Descriptive statistics for the sampling distribution of* $\hat{\sigma}^2$ *based on 50,000 Monte Carlo runs. Experimental model:* $X_t = b X_{t-2} \varepsilon_{t-1} + \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma^2)$ *with* $\sigma^2 = 1$.

b	*n*	*Mean*	*Med*	*SD*	*MAD*	*SK*	*RSK*	*Range*
0.0	200	1.004	1.000	0.123	0.153	0.216	0.031	1.007
	500	1.002	1.001	0.078	0.097	0.103	0.001	0.696
	1000	1.001	1.000	0.055	0.067	0.053	0.004	0.460
0.1	200	1.005	1.001	0.124	0.155	0.197	0.028	1.034
	500	1.002	1.001	0.079	0.098	0.104	0.008	0.652
	1000	1.001	1.000	0.055	0.068	0.058	0.011	0.463
0.2	200	1.008	1.002	0.128	0.158	0.207	0.036	1.062
	500	1.003	1.003	0.081	0.101	0.096	-0.001	0.690
	1000	1.001	1.001	0.057	0.071	0.054	0.009	0.514
0.3	200	1.014	1.008	0.134	0.166	0.217	0.035	1.140
	500	1.006	1.006	0.086	0.106	0.073	-0.002	0.747
	1000	1.003	1.002	0.061	0.076	0.022	0.012	0.586
0.4	200	1.026	1.020	0.146	0.179	0.228	0.035	1.320
	500	1.013	1.013	0.095	0.116	0.010	-0.004	0.868
	1000	1.007	1.007	0.070	0.085	-0.110	0.009	0.763
0.5	200	1.051	1.043	0.167	0.202	0.257	0.034	2.345
	500	1.029	1.029	0.113	0.134	-0.092	0.002	1.522
	1000	1.017	1.020	0.086	0.102	-0.360	-0.018	0.938
0.6	200	1.106	1.092	0.209	0.243	0.484	0.042	4.700
	500	1.069	1.069	0.145	0.170	-0.067	0.001	3.306
	1000	1.048	1.054	0.116	0.132	-0.392	-0.025	1.731

Legenda. *Med=Median, SD=Standard Deviation, MAD=Median Absolute Deviation, SK=Skewness, RSK=Resistant Skewness.*

Figure 1. *Distribution of* $\Delta v = Var^*(\hat{b}) - Var(\hat{b})$. $Var(\hat{b})$ *estimated by 50,000 Monte Carlo runs. Parametric bootstrap on the left and residual bootstrap on the right in each sub-panel.*

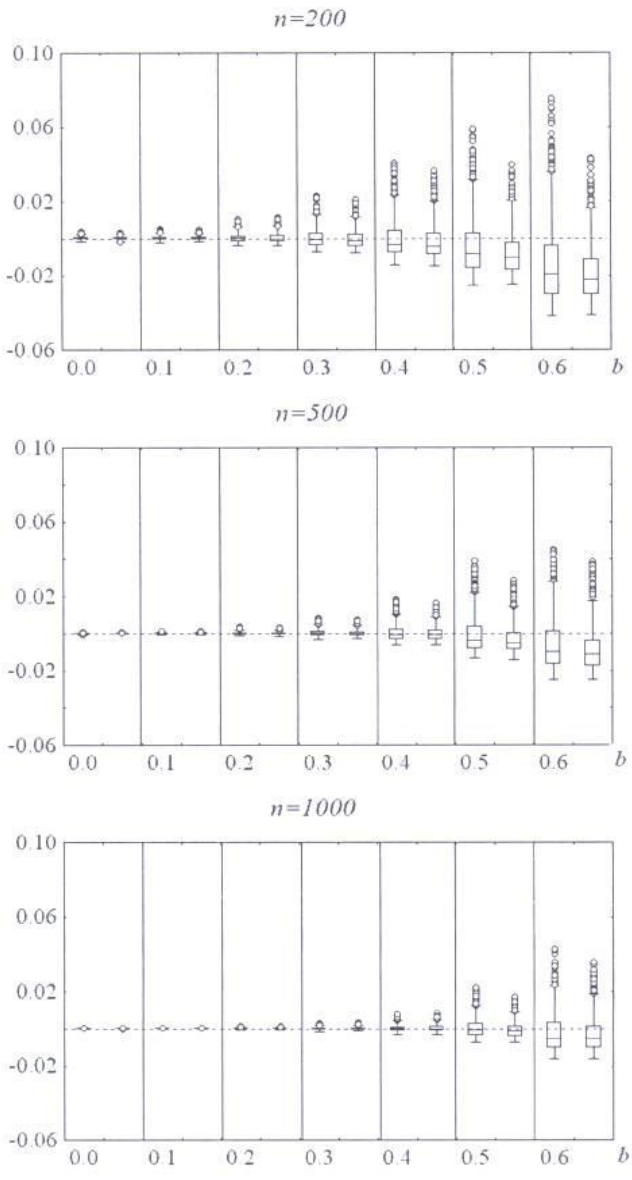

Figure 2. *Distribution of* $\Delta v = Var^*(\hat{\sigma}^2) - Var(\hat{\sigma}^2)$. $Var(\hat{\sigma}^2)$ *estimated by 50,000 Monte Carlo runs. Parametric bootstrap on the left and residual bootstrap on the right in each sub-panel.*

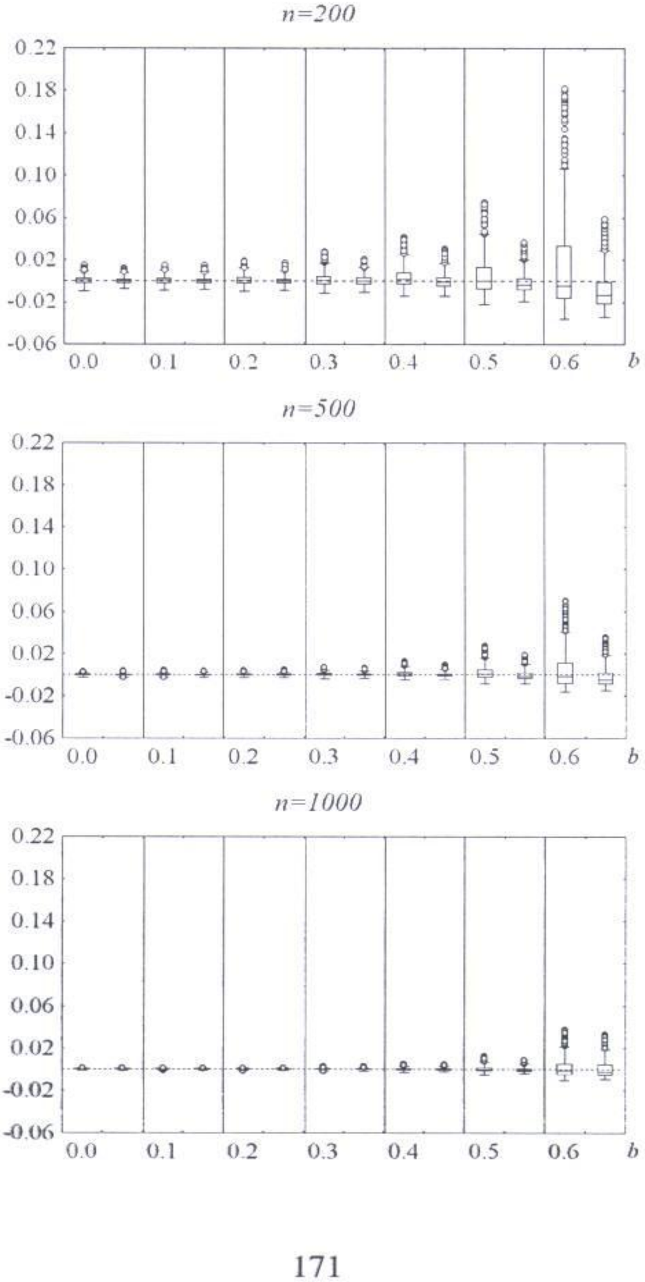

Table 3. *Empirical coverage for b and σ^2. Time series length n: 200, 500 and 1,000. Monte Carlo runs: 1,000. Bootstrap replicates 999. Nominal confidence level 0.90.*

b		**PB**			**RB**		
b	*Method*	*200*	*500*	*1000*	*200*	*500*	*1000*
0.0	NBCI	.933	.907	.901	.923	.910	.900
	BPCI	.916	.906	.898	.907	.904	.896
0.1	NBCI	.930	.903	.906	.914	.920	.908
	BPCI	.916	.899	.897	.896	.909	.904
0.2	NBCI	.921	.904	.903	.901	.916	.901
	BPCI	.909	.890	.889	.881	.911	.902
0.3	NBCI	.908	.906	.906	.926	.930	.907
	BPCI	.905	.892	.891	.931	.925	.897
0.4	NBCI	.905	.910	.909	.878	.891	.909
	BPCI	.917	.916	.904	.892	.902	.906
0.5	NBCI	.848	.895	.893	.825	.875	.887
	BPCI	.852	.908	.909	.836	.884	.899
0.6	NBCI	.773	.811	.839	.770	.802	.863
	BPCI	.763	.823	.854	.760	.804	.866

σ^2		**PB**			**RB**		
b	*Method*	*200*	*500*	*1000*	*200*	*500*	*1000*
0.0	NBCI	.881	.892	.890	.880	.895	.900
	BPCI	.887	.895	.889	.937	.955	.943
0.1	NBCI	.892	.891	.884	.899	.886	.897
	BPCI	.899	.894	.886	.935	.950	.943
0.2	NBCI	.896	.898	.876	.896	.894	.898
	BPCI	.896	.899	.874	.936	.962	.950
0.3	NBCI	.902	.899	.876	.901	.888	.897
	BPCI	.906	.892	.878	.943	.962	.966
0.4	NBCI	.910	.903	.903	.877	.909	.907
	BPCI	.885	.888	.900	.934	.961	.961
0.5	NBCI	.904	.902	.914	.877	.891	.889
	BPCI	.854	.868	.894	.933	.929	.933
0.6	NBCI	.905	.867	.891	.850	.825	.859
	BPCI	.715	.729	.792	.853	.845	.841

Table 4. *Median and MAD of CI lengths for b. Time series length n: 200, 500 and 1,000. Monte Carlo runs: 1,000. Bootstrap replicates: 999. Nominal confidence level 0.90.*

Median			**PB**			**RB**	
b	*Method*	*200*	*500*	*1000*	*200*	*500*	*1000*
0.0	NBCI	.2394	.1483	.1046	.2410	.1497	.1048
	BPCI	.2402	.1490	.1049	.2412	.1498	.1052
0.1	NBCI	.2495	.1565	.1109	.2488	.1570	.1110
	BPCI	.2478	.1568	.1110	.2483	.1574	.1114
0.2	NBCI	.2849	.1825	.1304	.2815	.1831	.1307
	BPCI	.2796	.1812	.1301	.2768	.1821	.1305
0.3	NBCI	.3414	.2283	.1648	.3375	.2253	.1657
	BPCI	.3282	.2218	.1626	.3245	.2200	.1621
0.4	NBCI	.4166	.2917	.2185	.4019	.2883	.2163
	BPCI	.3868	.2754	.2090	.3760	.2740	.2074
0.5	NBCI	.4896	.3806	.2999	.4683	.3591	.2844
	BPCI	.4446	.3457	.2726	.4297	.3299	.2626
0.6	NBCI	.5584	.4577	.3825	.5239	.4379	.3816
	BPCI	.4880	.4020	.3374	.4729	.3933	.3350

MAD			**PB**			**RB**	
b	*Method*	*200*	*500*	*1000*	*200*	*500*	*1000*
0.0	NBCI	.0214	.0081	.0049	.0205	.0074	.0043
	BPCI	.0209	.0088	.0052	.0199	.0081	.0050
0.1	NBCI	.0287	.0134	.0074	.0258	.0121	.0069
	BPCI	.0274	.0138	.0077	.0246	.0118	.0074
0.2	NBCI	.0520	.0267	.0141	.0514	.0249	.0140
	BPCI	.0463	.0250	.0136	.0468	.0232	.0138
0.3	NBCI	.0893	.0499	.0267	.0836	.0468	.0269
	BPCI	.0756	.0432	.0247	.0700	.0404	.0248
0.4	NBCI	.1355	.0844	.0498	.1275	.0832	.0488
	BPCI	.1133	.0700	.0421	.1023	.0690	.0421
0.5	NBCI	.1827	.1375	.0987	.1544	.1152	.0828
	BPCI	.1427	.1056	.0788	.1213	.0934	.0670
0.6	NBCI	.2259	.1783	.1570	.1728	.1576	.1402
	BPCI	.1617	.1358	.1187	.1350	.1214	.1058

Table 5. *Median and MAD of CI lengths for* σ^2. *Time series length n: 200, 500 and 1,000. Monte Carlo runs: 1,000. Bootstrap replicates: 999. Nominal confidence level 0.90.*

Median		**PB**			**RB**		
b	*Method*	*200*	*500*	*1000*	*200*	*500*	*1000*
0.0	NBCI	.4030	.2552	.1793	.3950	.2524	.1787
	BPCI	.4042	.2562	.1804	.3962	.2528	.1798
0.1	NBCI	.4060	.2573	.1811	.3981	.2541	.1807
	BPCI	.4075	.2586	.1821	.3994	.2552	.1813
0.2	NBCI	.4195	.2658	.1876	.4090	.2622	.1866
	BPCI	.4210	.2672	.1885	.4121	.2638	.1880
0.3	NBCI	.4438	.2836	.2014	.4308	.2782	.1997
	BPCI	.4466	.2840	.2024	.4316	.2796	.2012
0.4	NBCI	.4834	.3131	.2282	.4573	.2982	.2233
	BPCI	.4838	.3139	.2271	.4577	.3000	.2230
0.5	NBCI	.5328	.3682	.2779	.4974	.3395	.2611
	BPCI	.5308	.3670	.2748	.4992	.3404	.2587
0.6	NBCI	.6437	.4483	.3559	.5617	.4056	.3419
	BPCI	.6384	.4474	.3506	.5602	.4053	.3365

MAD		**PB**			**RB**		
b	*Method*	*200*	*500*	*1000*	*200*	*500*	*1000*
0.0	NBCI	.0603	.0252	.0138	.0550	.0227	.0133
	BPCI	.0609	.0269	.0144	.0559	.0234	.0140
0.1	NBCI	.0655	.0257	.0138	.0570	.0238	.0139
	BPCI	.0674	.0266	.0150	.0577	.0243	.0141
0.2	NBCI	.0753	.0282	.0155	.0631	.0269	.0154
	BPCI	.0764	.0282	.0161	.0640	.0273	.0157
0.3	NBCI	.0871	.0356	.0200	.0746	.0334	.0210
	BPCI	.0887	.0358	.0210	.0717	.0339	.0220
0.4	NBCI	.1107	.0544	.0335	.0928	.0450	.0317
	BPCI	.1126	.0555	.0322	.0932	.0440	.0309
0.5	NBCI	.1670	.0966	.0632	.1098	.0748	.0496
	BPCI	.1635	.0913	.0604	.1099	.0752	.0477
0.6	NBCI	.2680	.1646	.1271	.1621	.1138	.1082
	BPCI	.2544	.1641	.1254	.1573	.1141	.1071